Financing Capacity Investment Under Demand Uncertainty: An Optimal Contracting Approach

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We study the capacity choice problem of a firm whose access to capital is hampered by financial frictions in the form of moral hazard. The firm must therefore optimize not only its capacity investment under demand uncertainty, but also its sourcing of funds. Ours is the first study of this problem to follow an optimal contracting approach, where feasible source of funds are derived endogenously from fundamentals and include standard financial claims (debt, equity, convertible debt, call and put warrants, etc.) and combinations thereof. We characterize the optimal capacity level. First, we find conditions under which a feasible financial contract exists that achieves first-best. When no such contract exists, we find that under optimal financing, the choice of capacity sometimes exceeds strictly the efficient level. This runs counter to the literature on financing capacity investment in funds and the intuition that by raising the cost of external capital and hence the unit capacity cost, financial market frictions lower the optimal capacity level. We trace the value of increasing capacity beyond the efficient level to a *bonus effect* and a *demand differentiation effect.* In contrast to most of the literature on financing capacity, our results are robust to a change of financial contract.

Key words: Capacity, Optimal Contracts, Financial Constraints, Newsvendor Model.
History:

1. Introduction

Capacity investment decisions for new products or markets are often made under considerable demand uncertainty. When calibrating capacity, management must navigate between the Scylla of excess demand's opportunity cost and the Charybdis of excess capacity's cost. While firms investing in capacity need to mobilize human and physical capital, they must also avail of funds to acquire those. When cash is short, they have to tap outside markets for funds. External financing can come from a variety of sources: bank loans, trade credit, bonds, private equity, public stock markets, etc. Hence companies must not only calibrate their capacity investment and the corresponding funding amount, but also optimize their sourcing of funds.

This paper takes an optimal contracting approach to study the interplay between the operational and financial facets of capacity investment. We consider the capacity choice problem of a firm with limited cash and whose access to external funds is hampered by financial frictions, i.e., moral hazard. When calibrating its capacity investment, the firm must consider the sources of funds available. These are derived endogenously and include standard financial claims (debt, equity, convertible debt, warrants, etc.) and combinations thereof.

After characterizing the optimal source of funds to finance a given capacity level, we show that under optimal financing, the optimal capacity can exceed strictly the efficient level. Specifically, we find that the firm achieves first best if moral hazard is mild or if cash is high enough. Else, the optimal capacity level exceeds the efficient level and increases with the moral hazard problem's severity and decreases with the firm's internal funds. When the frictions are too strong and cash is too low, financing is impossible and capacity is set at zero, i.e, below the efficient level. Crucially, our approach guarantees that no change of financial contract can mitigate these costly deviations from first-best.

These results run counter to existing findings on financing capacity investments (e.g. Dada and Hu (2008), Alan and Gaur (2015)) and more generally to the intuition in finance that by raising the cost of external funds and hence the unit capacity cost, financial market frictions lower investment in capacity (e.g. Myers and Majluf (1984) or Jensen and Meckling (1976)). Conversly, by reducing the need for external funds, leading to lower unit cost, higher internal cash should increase capacity investment. That logic holds when financial frictions remain constant or perhaps increase with the amount raised. However, we identify two effects, the *bonus* and *demand differentiation* effects, whereby financial frictions may actually decrease as capacity increases.

Specifically, we consider a firm with limited cash making a capacity investment choice under demand uncertainty, which we formalize as a newsvendor model with two features.

First, the firm's owner has limited cash but can raise funds from a competitive investor (Section 7 studies a strategic investor). This feature alone causes no tangible departure from the standard newsvendor model. Indeed, absent financial frictions, Modigliani and Miller (1958)'s irrelevance theorem implies that the firm's problem is unaffected by the need for funds and that the choice of a source of funds is a matter of irrelevance. (Section 4 revisits the theorem for our model.)

Second, we assume that, by exerting a (non-contractible) costly effort, the firm can improve the distribution of demand in the sense of the Monotone Likelihood Ratio Property (MLRP). Effort can, for instance, stand for conducting a sales campaign, or improving a product's design. Loosely speaking, MLRP means that higher demand realizations are more indicative of high effort. The non-contractibility of effort, i.e., that it cannot be set by contract, constitutes a financial friction that renders the firm's need for funds and its optimal sourcing of funds relevant.

Given this, the firm must optimize not only its capacity but also its funding, i.e., the financial claim to offer the investor. We follow an optimal contracting approach in which agents optimize over feasible claims derived *endogenously from fundamentals*, i.e., preferences, physical constraints (e.g., production technology), and constraints on contractibility (i.e., which variables can be set in an enforceable contract). This contrasts with existing operations-and-finance studies that assume an exogenous set of feasible claims, typically debt or, more rarely, debt and equity.[1]

Simple assumptions about fundamentals ensure claims satisfy three conditions. All claims satisfying these are feasible sources of funds over which the firm optimizes. First, we assume that effort apart, all variables (e.g., capacity, funding, costs, revenues, etc.) are contractible bar one: demand. Revenues being contractible, and given they map one-to-one with demand when demand is below capacity, this amounts to assuming that only unmet demand is not contractible. Hence, financial contracts specify a capacity, an amount of funding by the investor, and a financial claim, i.e., the promise of repayments to the investor contingent on revenues. Second, we assume the firm to be protected against funds falling below zero. Hence claims must satisfy *limited liability*, i.e., repayments cannot exceed revenues. Third, we assume the firm can report artificially inflated revenues, and will do so if that leads to lower repayments. To avoid such manipulation, claims must be *monotonic*, i.e., repayments must increase with revenues.

The feasible set so-defined includes standard financial claims such as debt, equity, convertible debt, or warrants and combinations thereof (e.g., debt plus equity, etc.). For instance, a debt claim defines repayments equal to the realized revenue if below the debt's face value, and otherwise equal to that face value. Similarly, an equity stake defines repayments equal to a fixed fraction of the realized revenue. Both claims are feasible as the repayment functions only depend on revenues, are clearly non-decreasing and respect limited liability.

We first characterize the optimal source of funds to finance a given capacity. (Capacity being contractible, it is set independently from the financial claim, which is in contrast to the bulk of the operations-and-finance literature.) A priori, different sources of funds may have their appeal. For instance, equity may be suitable to control excessive risk-taking by the firm, a temptation debt might instead exacerbate. However, extending Innes (1990) to our context, we show that using

[1] In the literature, the optimal claim is typically chosen endogenously but from an exogenous set. Limiting a priori the feasible set, one runs the risk of excluding viable claims that could achieve first-best, i.e., one risks concluding that distortions of operations are needed when they could be avoided by a simple contract change. (see Section 2).

cash should be a priority and that if external funds are needed, debt financing dominates. Further, and in contrast to Innes (1990), first best effort can be achieved and the optimal debt does not always involve default risk.

Determining the optimal financing is a key methodological step. Indeed, the paper main's point is to characterize the firm's optimal capacity investment *given that investment is financed optimally*. Absent financial frictions, i.e., if effort is contractible, the optimal capacity is the standard newsvendor quantity. With mild enough frictions, the optimal capacity equals the efficient level. (This is true even when the debt claim involves default risk.) If the frictions are stronger, however, no feasible contract can achieve first-best and the optimal capacity level is strictly above the efficient level. Finally, when frictions are too strong, financing is impossible and the project is abandoned, i.e., capacity is set at zero, hence below the efficient level.

Our most interesting result is perhaps that intermediate frictions lead to over-investment. In that case, increasing capacity beyond the efficient level commits the firm to exerting effort, i.e., relaxes its incentive constraint via two channels. First, raising capacity above the efficient level increases the firm's expected payoff for demand realizations above the first best by a lump sum, which is akin to a bonus. This *bonus effect* enhances incentives. Second, the increase in payoff the extra capacity brings about is higher for higher demand realizations above the first best. This *demand differentiation effect* also boosts incentives because under MLRP, higher demand realizations are more indicative of effort. We show that neither effect can be achieved at first best capacity with a suitable financial claim. Indeed, such a claim would violate the monotonicity condition to get the bonus effect, and the non-contractibility of unmet demand to get the demand differentiation effect.

In short, when moral hazard is severe enough, the firm commits to exerting effort by setting capacity above the efficient level to exploit the incentive power of the *bonus* and *demand differentiation* effects. Moreover, as moral hazard becomes more severe, the firm will raise capacity further to increase both effects. Conversely, optimal capacity will decrease with the firm's internal funds.

The paper proceeds as follows. Section 2 presents a literature review. Section 3 introduces the model. Section 4 studies the contractible effort benchmark. Section 5 establishes the optimality of debt financing. Section 6 studies the optimal capacity level, the optimality of over-investing and the role of internal funds for capacity investment. Section 7 extends our results to a strategic investor. Section 8 concludes. All proofs are in the Appendix.

2. Related Scholarship

Our over-investment result contrasts with standard corporate finance theories (see Tirole 2006). Those generally assume financial frictions such as information asymmetry (Myers and Majluf 1984) or moral hazard (Jensen and Meckling 1976) causing external funds to be more costly than internal funds. Hence, firms needing more outside financing incur a higher cost of capital, and tend to invest less, not more, than the first-best level. Our over-investment result stems from the fact that in our model, higher capacity can reduce, not increase, the cost of external funds.

To be complete, over-investment arises in one corporate finance theory: Jensen (1986)'s free cash flow theory positing that investment-prone management employ cash for excessive investment. Anticipating management's bias, investors will not provide external funds over what is needed for optimal investment. Hence only firms with excess cash invest more than optimally. Our result is the opposite: only firms with insufficient cash invest more than the first-best level.

Our paper belongs to the nascent literature on the interplay between capacity choices and financial decisions. This line of research has derived a rich set of implications for how a firm's funding needs affect its capacity or technology choices and for how these in turn impact the firm's financial policy. For instance, Alan and Gaur (2015) consider a newsvendor seeking funds from a bank and find that capacity is set optimally below the efficient level. In Li et al. (2013)'s dynamic inventory model, optimal inventory and financial decisions are found to be myopic and increasing in inventory level and retained earnings. Boyabatli and Toktay (2011) consider a multi-product firm making capacity and (flexible or dedicated) technology choices and show that demand uncertainty's impact on those choices is affected by the firm's need for funds.

In this line of research, Dada and Hu (2008), the paper closest to ours, studies the problem of funding a capital-constrained newsvendor similar to ours and thus provides a relevant benchmark. The paper shows that assuming debt financing, *under-investment* arises. Our set-up differs from theirs by expanding the set of feasible sources of funds beyond debt to account for general financial claims. Further, we assume the existence of financial frictions, without which there is no problem in the first place. Indeed, absent frictions and if the feasible set of contracts is not limited exogenously, a financial contract achieving first best always exists, a point Dada and Hu (2008)'s Proposition 4 illustrates. In fact, in the context of Dada and Hu (2008), not only debt, but any type of claim (e.g., equity) can achieve first best (see our Proposition 5 and Modigliani and Miller (1958)).

More generally, these differences (i.e, financial frictions and a more general set of feasible financial contracts) set our work apart from the operations-and-finance literature. Indeed the papers above model financing in similar ways: an interest rate is set upfront and the newsvendor then chooses the loan size. In Dada and Hu (2008) the interest rate is set by a monopolist bank. In Buzacott and Zhang (2004) and Alan and Gaur (2015) too but the loan size is capped. In Boyabatli and

Toktay (2011), the rate is set by a competitive bank for each technology and loans may be secured or not. Li et al. (2013) focus on short-term debt with an exogenous interest rate.

These papers thus share two main exogenous restrictions on the set of feasible claims. First, they focus *a priori* on debt, ruling out other claims, e.g., equity, convertible debt, etc. They only allow *internal equity*, i.e., equity contributed by the newsvendor. Second, they assume *a priori* that financing terms (i.e. interest rates) are *not* conditional on the funding (i.e. loan size).[2] Under these restrictions, the financial claim typically leads the firm to distort its capacity decision away from first best. Yet, it is possible that within these models, claims other than debt improve the outcome perhaps even to first best. Moreover, even restricting to debt financing, the same might hold for loan size-dependent interest rates (see Dada and Hu (2008)'s Proposition 4).

Thus, our paper departs from this literature by adopting an optimal contracting approach that is now the standard methodology in corporate finance (e.g. Tirole 2006). In this approach, agents optimize over a set of feasible contracts derived endogenously from assumptions about fundamentals, i.e. preferences, physical constraints, and contractibility. Further, we specify explicitly the frictions that make the financing problem relevant to operations decisions (i.e., ensuring the Modigliani and Miller irrelevance theorem does not hold). We then explore the conditions under which deviations from efficient outcomes arise, but only once feasible contractual solutions are exhausted. This approach ensures that when the firm deviates from first best, no feasible contract can alleviate the firm's incentive to do so. In this sense, our over-investment result is robust.

Technically, our paper builds on the principal-agent literature with risk-neutrality and limited liability (e.g., Oyer 2000, Gromb and Martimort 2007, Poblete and Spulber 2012). In particular, Dai and Jerath (2013, 2016) study capacity choice models in which a sales agent must be induced by her wage contract to take a demand-enhancing action. Dai and Jerath (2013) show that capacity above the first best may be optimal for a reason akin (but not identical) to our *demand differentiation effect*. However, the salesforce compensation problem they study is quite different from our financing problem. In particular, wage contracts are not constrained by the requirements of financial contracts (notably the monotonicity condition). Moreover, moral hazard makes their sales agent better off whereas it makes our firm worse off. As a result, their optimal wage contract is never optimal in our set-up and vice versa, our *bonus effect* never arises in theirs and the financial issues we study (e.g. how cash affects investment) cannot be addressed with their model.

[2] That is, the investor does not offer a menu of interest rates associated with different loan sizes for the firm to choose from. Instead, he offers a fixed rate, to which the firm responds by choosing the loan size.

3. A Model of Capacity Investment Financing

3.1. The Newsvendor Model with Financing under Moral Hazard

We study the problem of a firm with limited cash making a capacity investment choice under demand uncertainty. We formalize this situation as a newsvendor model, with two features: the firm's owner has limited wealth but can raise funds externaly and he can affect the distribution of demand by taking a non-contractible action. It is worth noting that limited wealth *per se* (see Section 4), or the non-contractibility of the action *per se* (see Section 5) causes no meaningful departure from the standard newsvendor model. However, the combination of both features does.

Newsvendor model Assume universal risk-neutrality and no discounting.[3] Consider a firm whose sole owner (henceforth "the firm") has an investment project. The firm faces stochastic demand D_1 with distribution $f_1(\cdot)$, cumulative distribution $F_1(\cdot)$ and complementary cumulative distribution $\bar{F}_1(\cdot) \equiv 1 - F_1(\cdot)$. To simplify, we assume that f_1 is strictly positive over $\mathbb{R}_+$. The firm chooses a capacity $q \in \mathbb{R}_+$ at unit cost $c > 0$ before demand is realized, and eventually receives a revenue $r > c$ per unit sold and salvage value $s < c$ per unsold unit.

Effort choice If the firm does not set up any capacity (i.e., if $q = 0$), we say it abandons the project. Otherwise, once capacity $q > 0$ is installed but before demand is realized, the firm can opt to run the project diligently (e.g., conduct a suitable sales campaign, improve product design, etc.) or not, which we model as an effort choice. If the firm works (effort $e = 1$), it incurs a non-monetary cost $\kappa_1 > 0$. This can be a labor cost for the owner, the opportunity cost of allocating attention l away from other projects, etc.[4] If instead the firm shirks (effort $e = 0$) it incurs a smaller non-monetary cost $\kappa_0 > 0$, i.e., with $\Delta\kappa \equiv \kappa_1 - \kappa_0 > 0$.

A priori, the firm could to adjust the project to the lower level of effort (e.g., set up cheaper or costlier capacity, set a different product price, etc.), which might affect demand. To simplify, we assume that irrespective of such adjustments, the project's value is always negative if the firm shirks, so that abandoning it ($q = 0$) is preferable.

A case of particular interest is when the firm shirks ($e = 0$) but does not adjust the different project's features relative to when it works ($e = 1$), i.e., unit price, cost and salvage value remain equal to r, c and s.[5] In this case, shirking shifts demand to D_0 with a distribution f_0 less favorable than f_1 in the sense of the Monotone Likelihood Ratio Property (MLRP), i.e., f_1/f_0 is strictly

[3] Risk-neutrality is standard and important for the optimality of debt financing. No-discounting is only for simplicity.

[4] The assumption that no cost is incurred if $q = 0$ is for simplicity. It amounts to assuming that κ_0 is a fixed cost which will allow us to focus on cases where the firm exerts $e = 1$ or sets $q = 0$.

[5] Even absent this assumption, the firm would have incentive not to perform any such adjustment as this would reveal it is shirking.

increasing over $\mathbb{R}_+$. Loosely speaking, MLRP means that higher demand realizations are more indicative of high effort.[6] Note also that MLRP implies but is stronger than first-order stochastic dominance and increasing hazard ratios.

First-Best For given capacity $q \geq 0$ and effort $e \in \{0,1\}$, the project's monetary payoff is randomly distributed over $[sq, rq]$ according as per density $g_{e,q}$ (implied by density f_e) and is

$$P_{e,q} \equiv sq + (r-s)(D_e \wedge q). \tag{1}$$

Denote the expected profit in the standard newsvendor model as

$$\pi_e(q) \equiv \mathbb{E}[P_{e,q}] - cq \tag{2}$$

As per standard arguments, $\pi_e(\cdot)$ is strictly concave over $\mathbb{R}_+$ and the first-best optimal capacity is

$$q_e^{FB} \equiv \arg\max_{q \in \mathbb{R}_+} \pi_e(q) = F_e^{-1}\left(\frac{r-c}{r-s}\right) \tag{3}$$

We assume that given r, c and s, the project is viable if $e=1$ but not if $e=0$, i.e.,

$$\max_{q \in \mathbb{R}_+} \pi_1(q) - \kappa_1 > 0 > \max_{q \in \mathbb{R}_+} \pi_0(q) - \kappa_0 \tag{4}$$

so that the first-best optimal outcome is $e=1$ and q_1^{FB}.

Financial contracts The firm has cash $W \geq 0$. If this is not enough to fund its desired capacity, the firm can raise funds from a competitive investor and a financial contract. Such a contract specifies a capacity $q \geq 0$ to be set up, an investment $I \geq 0$ by the investor against a financial claim, i.e., the promise of a repayment contingent on variables that are contractible, i.e., that can be set in an enforceable contract. Note that, in contrast to much of the operations-and-finance literature, capacity q and funding amount I are set by the financial contract, not left for the firm to decide post-contracting.

Assumptions about fundamentals ensure that claims satisfy three conditions. First, we assume that all variables (e.g., capacity q, funding I, cost c, revenue $P_{e,q}$, etc.) are contractible except for effort e and demand D_e. (Section 4 studies contractible effort.) For instance, costs and revenues can typically be retrieved by auditing the firm. Effort represents less tangible inputs, like whether the firm's "best people" are allocated to the project, the intensity of a sales effort, or how much thinking goes into product design. As for demand, revenues $P_{e,q}$ being contractible and given they

[6] Following a demand realization d, an agent would revise his prior $\nu = \Pr[e=1]$ to posterior is $\frac{\Pr[e=1 \cap d]}{\Pr[d]} = \frac{\nu f_1(d)}{\nu f_1(d) + (1-\nu) f_0(d)} = \frac{1}{1 + \frac{(1-\nu)}{\nu} \frac{f_0(d)}{f_1(d)}}$ which, by MLRP, is strictly increasing with d.

map one-to-one with demand when demand is below capacity, demand's non-contractibility boils down to only unmet demand being non-contractible. This assumption seems reasonable as unfilled demands are typically difficult to observe. In our context, this implies that financial contracts cannot distinguish between realizations of demand D_e above capacity q as they all yield the same revenue rq. Hence financial contracts specify a capacity, an amount of funds contributed by the investor and repayment function $R(\cdot):[sq,rq]\mapsto\mathbb{R}$ such that given payoff $p\in[sq,rq]$, the repayment to the investor is $R(p)$, leaving the firm with net payoff $p-R(p)$.

Second, we assume that the firm's *limited liability* must be preserved: repayments cannot exceed revenues, i.e., $\forall p\in[sq,rq]$, $R(p)\le p$.

Third, we assume that the firm is able to report artificially inflated revenues by raising funds secretly from a third party. The firm may opportunistically do so to lower its repayment to the investor if the financial contracts specifies a lower repayment for higher (reported) revenues. Avoiding such manipulation requires that financial contracts be *monotonic* in that $R(\cdot)$ must be non-decreasing over $[sq,rq]$. This assumption is standard in finance. In the context of the newsvendor model, inflating revenues amounts to using the funds raised secretly to purchase units for r, thereby inflating revenues artificially, before reselling the units at their salvage value s.[7]

All claims satisfying these conditions are deemed feasible. Note that all repayments $R(p)$ need not be positive, i.e., a claim can call the investor to pay the firm for some payoff realizations. Note also that borrowing is feasible as a debt contracts with face value K maps into $R(p)=p\wedge K$ which satisfies the limited liability and monotonicity conditions.[8] Equity financing is feasible too as a fraction α of equity maps into $R(p)=\alpha p$. Other claims (e.g., convertible debt, put and call warrants, etc.) are also feasible, are as some combinations of claims (e.g., debt plus equity, etc.).

Importantly, the firm optimizes over the entire set of feasible financial contracts determined endogenously based on assumptions about fundamentals, i.e., preferences, physical constraints, and contractibility. This is in contrast to studies in which the set of feasible financial contracts is exogenous and determined by *a priori* restrictions.

3.2. The Firm's Problem

The firm's problem is to choose capacity $q\ge 0$, the funds I raised from the investor and the contractual repayments $R(\cdot)$ to the investor to maximize its expected payoff

$$\max_{q,I,R(\cdot)} \mathbb{E}\left[P_{e,q}-R\left(P_{e,q}\right)\right]+W+I-cq-\kappa_e \tag{5}$$

[7] An alternative foundation for monotonic financial contracts may be that the investor could engage in sabotage and reduce the payoff. (See Innes (1990) for a discussion of both foundations.) Strictly speaking, neither foundation rules out non-monotonic claims, but would imply they are equivalent to monotonic claims.

[8] We ignore the distinction between interest and principal payments, which is irrelevant in our model.

where the first term is the project's expected payoff net of repayment to the investor.

This choice is made under a number of constraints. First, the investor should accept the contract. Given risk-neutrality and no discounting, this requires that its expected payoff be no less than its investment. Hence, the following investor participation constraint must be satisfied

$$\mathbb{E}\left[R\left(P_{e,q}\right)\right] \geq I \tag{6}$$

Similarly, the firm should also accept the contract, which requires that its expected payoff be no less than its payoff under the status quo when the project is abandoned, i.e., the following firm participation constraint must be satisfied

$$\text{if } q > 0, \qquad \mathbb{E}\left[P_{e,q} - R\left(P_{e,q}\right)\right] + W + I - cq - \kappa_e \geq W \tag{7}$$

Second, taken together, the firm's cash and the funds raised from the investor should cover the cost of setting up capacity, i.e., the following funding constraint must be satisfied

$$I + W \geq cq \tag{8}$$

Third, the financial claim must be feasible, i.e., the following limited liability and monotonicity constraints must be satisfied

$$\forall\left(p, p'\right) \in [sq, rq]^2 \text{ with } p > p', \quad R(p) \leq p \quad \text{and} \quad R\left(p'\right) \leq R(p) \tag{9}$$

Finally, effort being non-contractible, it cannot be set as part of the financial contract. Instead, the firm chooses its level based on post-financing incentives. Therefore, unless the project is abandoned, the firm must prefer exerting the assumed effort level e rather than the alternative effort level $(1-e)$, i.e., the following incentive compatibility constraint must be satisfied:

$$\text{if } q > 0, \quad \mathbb{E}\left[P_{e,q} - R\left(P_{e,q}\right)\right] + W + I - cq - \kappa_e \geq \mathbb{E}\left[P_{(1-e),q} - R\left(P_{(1-e),q}\right)\right] + W + I - cq - \kappa_{(1-e)} \tag{10}$$

The previous problem can be simplified. First, notice that I, the amount raised from the investor, increases the firm's objective (5) and is bounded from above only by the investor's participation constraint (6) which must therefore be binding, i.e.,

$$\mathbb{E}\left[R\left(P_{e,q}\right)\right] = I \tag{11}$$

an expression we can use to eliminate I from the objective and the constraints.

Second, using definition (2) and eliminating constant W, objective (5) can be rewritten as

$$\max_{q, R(\cdot)} \quad \pi_e(q) - \kappa_e \tag{12}$$

Hence the firm's objective is to maximize the project's expected profit net of effort cost. This simply reflects the fact that the investor being competitive, its expected profit must equal zero and so the firm captures the project's full value.

After similar simplifications, the firm's participation constraint (7) can be written as

$$\text{if } q > 0, \qquad \pi_e(q) - \kappa_e \geq 0 \tag{13}$$

which given condition (4), cannot hold for $e = 0$ and holds for $e = 1$ only for $q > 0$ such that

$$q \leq \bar{q}_1 \equiv \max\{q \text{ s.t. } \pi_1(q) - \kappa_1 = 0\} \tag{14}$$

Note that $\bar{q}_1 > q_1^{FB}$.

This implies that optimization can be restricted to the following: either the firm sets up capacity $q \in (0, \bar{q}_1]$ and exerts $e = 1$, or the firm abandons the project ($q = 0$). In other words, incentive compatibility condition (10) should in fact state that the firm should exert effort $e = 1$ if $q > 0$.

Overall, the firm's problem can be written as follows:

$$\max_{q, R(\cdot)} \pi_1(q) \tag{15}$$

$$s.t.$$

$$q \in (0, \bar{q}_1] \tag{16}$$

$$\mathbb{E}[R(P_{1,q})] \geq (cq - W)^+ \tag{17}$$

$$\forall (p, p') \in [sq, rq]^2 \text{ with } p > p', \quad R(p) \leq p \quad \text{and} \quad R(p') \leq R(p) \tag{18}$$

$$\mathbb{E}[P_{1,q} - R(P_{1,q})] - \mathbb{E}[P_{0,q} - R(P_{0,q})] - \Delta\kappa \geq 0 \tag{19}$$

and if the previous problem is not feasible then the firm abandons the project ($q = 0$). (Note that (16) is the firm's participation constraint, (17) the funding constraint, (18) the constraint that the contract be feasible, and (19) the incentive compatibility constraint).

4. Frictionless Financing

This section explores briefly the frictionless case. In our model, this means effort is contractible. Similar cases have been studied in the operations-and-finance literature, albeit for restricted sets of feasible financial contracts (see, e.g. Xu and Birge 2004).

If effort is contractible, Modigliani and Miller (1958)'s irrelevance theorem holds and implies the firm's need to raise funds and the financial claim employed to be irrelevant. Indeed, since the contract can specify not only revenue-contingent repayments $R(\cdot)$ but also the effort level, the firm's problem is as before but without incentive compatibility constraint (19) and with e now an optimization variable, i.e., the firm maximizes its objective $\pi_1(q)$ by choosing q, $R(\cdot)$ and e.

Since repayments $R(\cdot)$ do not appear in objective (15) and incentive compatibility condition (19) is absent, effort e can be set independently from repayments $R(\cdot)$. This implies that the firm is indifferent between *all* repayment functions $R(\cdot)$ satisfying the remaining constraints. This is simply the Modigliani-Miller Theorem in our model's context.

LEMMA 1. *If effort is contractible, for a given capacity $q \in (0, \bar{q}_1]$, the firm is indifferent between all financial claims satisfying funding constraint (17) and feasibility constraint (18).*

Note that for all $q \in (0, \bar{q}_1]$, such financial claims exist. Indeed, consider the claim such that $\forall p \in [sq, rq]$, $R(p) = p$, i.e., selling the project to the investor.[9] Clearly, this claim is feasible, i.e., satisfies condition (18). It satisfies funding condition (17) as, by definition of $R(\cdot)$ and $\bar{q}_1$

$$\mathbb{E}[R(P_{1,q})] = \mathbb{E}[P_{1,q}] \geq cq \tag{20}$$

Hence, all capacities $q \in (0, \bar{q}_1]$ being "fundable" by feasible claims, the problem reduces to

$$\max_{q \in (0, \bar{q}_1]} \pi_1(q) \tag{21}$$

That is, with contractible effort, the firm's problem amounts to the standard newsvendor problem, hence the following result.[10]

PROPOSITION 1. *If effort is contractible, the firm's optimal decisions are to exert effort $e = 1$ and set capacity at the corresponding first-best level q_1^{FB}.*

Therefore, we can solve the firm's problem as if cash W were sufficient to fund the chosen capacity. In a sense, Proposition 1 justifies the use in the literature of the standard newsvendor model. Indeed, despite the firm's need for funds, the analysis boils down to the standard model. Absent financial frictions, the need to raise funds has *per se* no impact.

By the same token, absent financial frictions, any distortion in capacity level caused by the need for funds is never robust to a contract change. This is illustrated by Dada and Hu (2008) who show that the under-investment decision they characterize for fixed-rate debt contracts disappears when the interest rate can be contingent on the loan size. By contrast, our paper specifies a friction (i.e. moral hazard) and optimizes over feasible claims, and so no contract change can alleviate the deviations from first-best we derive below.

[9] Note that although the firm retains no economic interest in the project, it is compensated by the investor's investment $I = \mathbb{E}[P_{1,q}]$, which exceeds the funding needs $(cq - W)^+$.

[10] Xu and Birge (2004)'s Proposition 2.1 also applies the Modigliani-Miller Theorem to the newsvendor model albeit restricting attention to debt claims. Our result simply uses the fact that the theorem applies to all claims. In both papers, these are obviously only benchmark results.

5. The Optimality of Debt Financing

Effort is now determined by the firm's incentives *after* financing, as per incentive compatibility constraint (19). In other words, effort's non-contractibility causes a friction in the firm's access to funds and the Modigliani-Miller Theorem no longer holds. This raises the question of the optimal source of funds for a given capacity q and its influence on the firm's choice of capacity. To address this problem, we adapt Innes (1990)'s optimal contracting approach to our context. This guarantees that the over-investment result of Section 6, our main result, is robust to any financial contract change. This is in contrast to the literature on the capacity investment problem, which typically omits optimizing over the feasible financial claims.

Specifically, we show that in the newsvendor model, MLRP extends from demands to payoffs (see Lemma 4 in Appendix A). MLRP is important for the incentive constraint. In short, MLRP implies that holding expected repayments constant, claims with higher repayments for higher payoffs are more detrimental to incentives to exert effort (see Lemma 5 in Appendix A). But a debt claim precisely limits repayments for higher revenue realizations by maximising repayments for lower realizations. We show next that for a given capacity q, the optimal claim is indeed a debt claim, which we characterize.

PROPOSITION 2. *For a given $q \in (0, \bar{q}_1]$, if claims satisfying conditions (17), (18) and (19) exist, the firm exerts effort $e = 1$ and an optimal financing of capacity cost cq is to use cash W and fund any short-fall $I(q) = (cq - W)^+$ with debt with face value $K(q)$, the unique solution to*

$$K - (r - s) \int_0^{\left(\frac{K-sq}{r-s}\right)^+} F_1(x)\, dx = (cq - W)^+ \tag{22}$$

- *If $W \geq cq$ then $K = 0$ and the firm does not raise funds.*
- *If $cq > W \geq (c - s) q$ then $K = I(q) \in (0, sq]$ and debt is riskfree.*
- *If $(c - s) q > W$ then $K \in (sq, rq]$ and debt involves default risk.*

Hence, $R(p) = p \wedge K(q)$ is an optimal repayment function. Equation (22) corresponds to funding condition (17) and investor's binding participation constraint (11), i.e. $\mathbb{E}[R(P_{1,q})] = (cq - W)^+$.

Note that for $q \in (0, \bar{q}_1]$, equation (22) has a unique solution. Indeed, its left-hand side, equal $\mathbb{E}[P_{1,q} \wedge K]$, is continuous and strictly increasing in K over $[sq, rq]$. For $K = 0$, $\mathbb{E}[P_{1,q} \wedge K]$ equals zero which is less than $(cq - W)^+$. For $K = rq$, $\mathbb{E}[P_{1,q} \wedge K]$ equals to $\mathbb{E}[P_{1,q}]$ which from condition (4), exceeds $(cq + \kappa_1)$ for $q \in (0, \bar{q}_1]$. This ensures existence and uniqueness of a solution. Said differently, the firm pays its debt with face value $K(q)$ in full if demand exceeds

$$\hat{d}(q) \equiv \left(\frac{K(q) - sq}{r - s}\right)^+ \tag{23}$$

Demand threshold $\hat{d}(q)$ is sometimes null even when the face value is positive. The optimal debt claim is then risk-free, which never occurs in Innes (1990). Similarly, the optimal debt claim never induces first-best effort in Innes' set-up, while it can in our context (see Section 6).

Finally, when the firm's cash suffices to fund the first-best optimal capacity, i.e., $W \geq cq_1^{FB}$, the claim $R(\cdot) \equiv 0$ (or, as per Proposition 2, $K = 0$) is optimal. Indeed, all constraints hold and the objective is maximized. With the results of Section 4, this illustrates that limited wealth *per se*, or the non-contractibility of effort *per se* causes no meaningful departure from the standard newsvendor model. Instead, their combination does as we now show.

6. Optimal Investment Decisions

6.1. Optimal Capacity

We now determine the optimal capacity choice q^* solving problem (5). For a given capacity q, it may be that no contract satisfies (17), (18) and (19), which is equivalent to incentive compatibility condition (19) being violated if R is the debt claim with face value $K(q)$. Noting that for $e = 0, 1$,

$$\mathbb{E}\left[P_{e,q} - P_{e,q} \wedge K(q)\right] - \kappa_e = (rq - K(q)) - (r - s) \int_{\hat{d}(q)}^{q} F_e(x)\, dx - \kappa_e \tag{24}$$

we can rewrite incentive compatibility constraint (19) as

$$L(q, \Delta\kappa) \equiv \int_{\hat{d}(q)}^{q} (F_0(x) - F_1(x))\, dx - \frac{\Delta\kappa}{(r - s)} \geq 0 \tag{25}$$

Recall that face value $K(q)$ affects the choice of efforts via the default-threshold for demand, $\hat{d}(q)$. Note that $L(q, \Delta\kappa)$ is decreasing in $\Delta\kappa$: the larger the extra cost of effort is, the more tempting is shirking. However, it is not clear how $L(q, \Delta\kappa)$ varies with q. Define $q_1^{\max}$ as the capacity in $[0, \bar{q}_1]$ for which $L(\cdot, \Delta\kappa)$ is maximum. For this capacity, the firm's incentive to work is strongest. It can be shown to satisfy the following.

Lemma 2. *Defining* $q_1^{\max} \equiv \arg\max_{q \in [0, \bar{q}_1]} L(\cdot, \Delta\kappa)$, *we have* $q_1^{\max} > q_1^{FB}$.

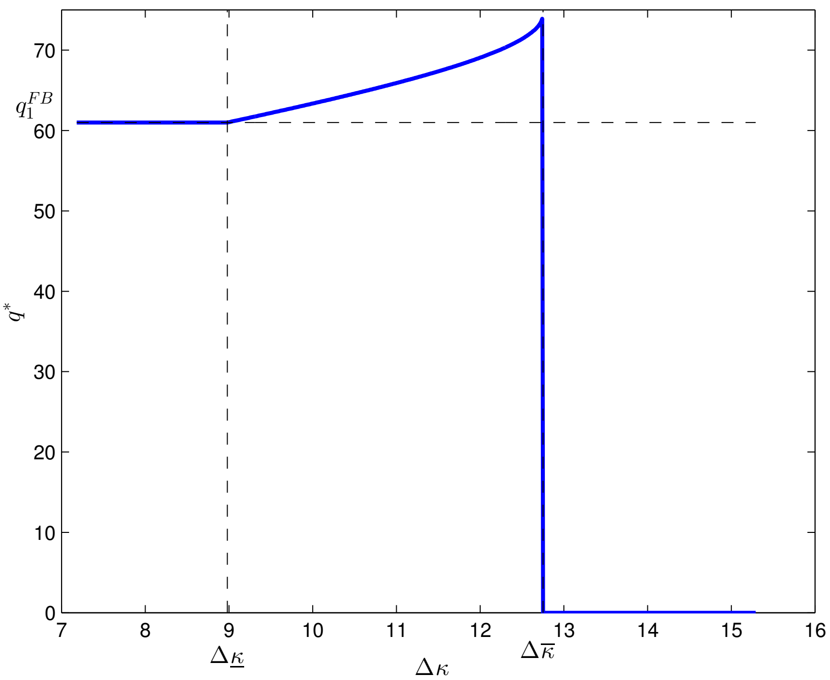
This key result means that increasing capacity beyond the first best level can boost incentives, i.e. relax incentive compatibility constraint (25). In Section 6.2, we formally trace this to two distinct mechanisms generated by a capacity increase: a *bonus effect* and a *demand differentiation effect.*

From the previous analysis, define

$$S(\Delta\kappa) \equiv \left\{ q \in \left[q_1^{FB}, q_1^{\max}\right] \text{ s.t. } L(q, \Delta\kappa) \geq 0 \right\} \tag{26}$$

the firm's overall problem boils down to the following. If $S(\Delta\kappa) \neq \varnothing$, the optimal capacity choice is $q^* = \inf S(\Delta\kappa)$. Indeed, objective (15) is strictly decreasing for $q \geq q_1^{FB}$. If instead $S(\Delta\kappa) = \varnothing$, the firm cannot both fund a capacity $q \in [q_1^{FB}, q_1^{\max}]$ and retains incentives to exert $e = 1$. In that case, condition (4) implies it optimally abandons the project ($q^* = 0$). This leads to our main result.

Figure 1 Optimal Capacity q^* as a function of moral hazard problem's severity $\Delta\kappa$



Note. $W = 0$, $r_1 = 11$, $c_1 = 10$, $s_1 = 0$, $D_1 \sim Gamma(10, 10)$ and $D_0 \sim Gamma(8, 10)$.

Theorem 1. *The firm's optimal choice of capacity q^* and effort level e^* solutions to problem (15) are as follows. If the firm undertakes the project ($q^* > 0$) then it exerts effort $e^* = 1$.*

- *If $W \geq cq_1^{FB}$, the firm funds the first-best capacity $q^* = q_1^{FB}$ with cash.*
- *If $cq_1^{FB} > W \geq (c - s)\, q_1^{FB}$, the firm funds the first-best capacity $q^* = q_1^{FB}$ with cash and riskless debt.*
- *If $(c - s)\, q_1^{FB} > W$, two thresholds $\Delta\underline{\kappa}$ and $\Delta\bar{\kappa}$ exist such that:*

(i) If $0 \leq \Delta\kappa \leq \Delta\underline{\kappa}$, the firm funds the first-best capacity $q^* = q_1^{FB}$ with cash and riskless debt.

(ii) If $\Delta\underline{\kappa} < \Delta\kappa \leq \Delta\bar{\kappa}$, the firm funds a capacity strictly above the first-best level ($q^* > q_1^{FB}$) with cash and risky debt. Further, q^* increases monotonically with $\Delta\kappa$.

(iii) If $\Delta\kappa > \Delta\bar{\kappa}$, the firm abandons the project ($q^* = 0$).

Figure 1 displays differential effort cost $\Delta\kappa$'s impact on optimal capacity q^* for $W = 0$, $r = 11$, $c = 10$, $s = 0$, and for demands D_1 and D_0 following Gamma distributions with shape and scale parameters $(10, 10)$ and $(8, 10)$ (yielding means of 100 and 80, and standard deviations of 31.6 and 28.28). Theorem 1 applies as Gamma distributions satisfy MLRP with respect to the shape parameter, holding the scale parameter constant. As the figure shows, the optimal capacity equals the first-best capacity $q^{FB} = 61$ as long as $\Delta\kappa$ is below $\Delta\underline{\kappa} = 8.98$. It is then strictly increasing above the efficient capacity level, until $\Delta\kappa$ reaches $\Delta\bar{\kappa} = 12.75$, beyond which the project is abandonned.

The intuition for this result is as follows. If possible, the firm sets up first-best capacity q_1^{FB} and works ($e^{FB} = 1$) to maximize objective (15). With enough cash ($W \geq (c-s)\, q_1^{FB}$), it can fund the first-best capacity with cash and risk-free debt, thus maintaining its incentives to work.[11] Suppose now that cash is lower. For $\Delta\kappa = 0$, incentive compatibility constraint (25) holds for any q and so the firm sets up q_1^{FB}. As $\Delta\kappa$ increases, the constraint tightens until it binds, which occurs for threshold $\Delta\underline{\kappa}$, but the firm sticks to q_1^{FB}. Beyond that point, the constraint is violated for q_1^{FB} but holds for some capacities in $(q_1^{FB}, q_1^{\max}]$. Objective (15) being strictly decreasing in q over that interval, the firm will set up the smallest such capacity, i.e., $q^* = \inf S(\Delta\kappa)$. As $\Delta\kappa$ increases, the firm must increase capacity q further beyond q_1^{FB} to retain incentives to work, i.e., q^* increases with $\Delta\kappa$. Finally, for some threshold $\Delta\bar{\kappa}$, constraint (25) is binding for $q = q_1^{\max}$. Beyond that point, incentive compatibility cannot be satisfied and the firm must abandon the project.

6.2. Optimal Over-investment: The Bonus and Demand Differentiation Effects

That the project is abandoned for $\Delta\kappa$ large enough and the first best obtains for $\Delta\kappa$ small enough is hardly surprising: both results would also arise in a fixed-capacity model. More interesting is that for intermediate values of $\Delta\kappa$, a capacity exceeding the first best level is optimal. As per Theorem 1, it is so when internal funds are low enough and moral hazard severe enough.

Corollary 1. *If $q^* > 0$, then $q^* \geq q_1^{FB}$ with $q^* > q_1^{FB} \Leftrightarrow W < (c-s)q_1^{FB}$ and $\Delta\kappa > \Delta\underline{\kappa}$.*

The firm sets up the first best capacity as long as $\Delta\kappa$ is small enough to be compatible with its incentive to exert effort, i.e., as long as constraint (25) holds for $q = q_1^{FB}$. When $\Delta\kappa$ exceeds threshold $\Delta\underline{\kappa}$, i.e., when $L(q_1^{FB}, \Delta\kappa) < 0$, the firm must raise q above q_1^{FB} as this relaxes constraint (25), i.e., increases $L(q, \Delta\kappa)$. We now identify two distinct channels through which this occurs.

Consider a capacity increase from q_1^{FB} to $q > q_1^{FB}$. For $e \in \{0,1\}$, the project's payoff (1) increases by $\Delta P_{e,q} = P_{e,q} - P_{e,q_1^{FB}}$. Focusing for now on demand realizations above q_1^{FB}, we decompose the expected payoff increase into two elements:

$$\mathbb{E}\left[\Delta P_{e,q} | D_e \geq q_1^{FB}\right] = b\,(q) + a_e\,(q) \tag{27}$$

where the payoff increase's mean for $e = 1$ and its expected deviation from that mean are denoted

$$b\,(q) \equiv \mathbb{E}\left[\Delta P_{1,q} | D_1 \geq q_1^{FB}\right] \quad \text{and} \quad a_e\,(q) \equiv \mathbb{E}\left[\Delta P_{e,q} - b(q) | D_e \geq q_1^{FB}\right].$$

[11] In that case, $K < sq$ so that $\hat{d} = 0$. This means that (25) is implied by (4).

The first term is akin to a fixed bonus $b(q)$ paid to the firm if demand exceeds q_1^{FB}. We refer to its impact on constraint (25) as the *bonus effect.* Note that this bonus being the same for all demand realizations above q_1^{FB}, implementing it would not require being able to tell these apart. The second term reflects deviations from $b(q)$. These differ across demand realizations which is only allowed by the fact that $q > q_1^{FB}$ allows the firm to meet otherwise unmet demand realizations in $(q_1^{FB}, q]$. We thus call this term's impact on constraint (25) the *demand differentiation effect.*

PROPOSITION 3. *For capacity $q \geq q_1^{FB}$, the bonus and demand differentiation effects are captured by $\beta(q) \equiv b(q) \cdot \left(\bar{F}_1(q_1^{FB}) - \bar{F}_0(q_1^{FB})\right)$ and $\alpha(q) \equiv a_1(q)\bar{F}_1(q_1^{FB}) - a_0(q)\bar{F}_0(q_1^{FB})$. We have*

$$L(q, \Delta\kappa) - L\left(q_1^{FB}, \Delta\kappa\right) = [\beta(q) + \alpha(q) - \varphi(q)] / (r - s)$$

with $\beta(q)$, $\alpha(q)$ and $\varphi(q)$ equal to zero at q_1^{FB} and strictly increasing over $(q_1^{FB}, +\infty)$.

At first-best capacity level $q = q_1^{FB}$, the bonus effect is first-order (i.e. $\partial\beta(q_1^{FB})/\partial q > 0$), and the demand differentiation effect second-order (i.e. $\partial\alpha(q_1^{FB})/\partial q = 0$).

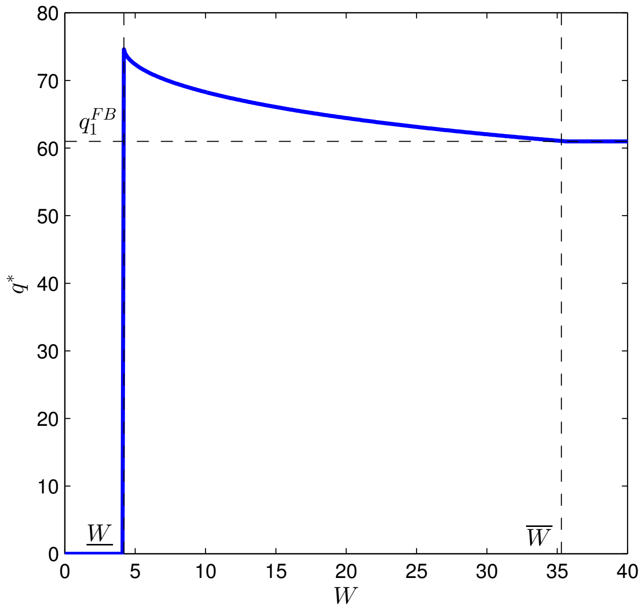
The *bonus effect* is captured by $\beta(q)$. Indeed, the incentive impact of bonus $b(q)$ is via the higher likelihood of getting it by working rather than shirking, i.e., $\bar{F}_1(q_1^{FB}) - \bar{F}_0(q_1^{FB})$. The *demand differentiation effect* is captured by $\alpha(q)$ as the incentive impact of deviations $\Delta P_{e,q} - b(q)$ from $b(q)$ is again via the difference between $e = 1$ and $e = 0$. The term $\varphi(q)$ captures the adverse incentive effect of increasing capacity, namely the higher salvage value for low demands (i.e., below q_1^{FB}), and the higher debt needed to fund the extra capacity. Given this, raising q above q_1^{FB} can boost incentives (i.e., $L(q, \Delta\kappa) > L(q_1^{FB}, \Delta\kappa)$) only if $\beta(q) > 0$ or $\alpha(q) > 0$. In fact, both $\beta(\cdot)$ and $\alpha(\cdot)$ are strictly positive and strictly increasing. At the first-best capacity level, constraint (25) is relaxed only by the bonus effect as the demand differentiation effect is negligible.

The bonus effect stems from the condition that financial claims be monotonic.[12] Indeed, to avoid distorting capacity away from its first-best level, the firm would favor adjusting the financial claim to include a bonus $b(q)$, but this is impossible. To see why, assume $s = 0$ for simplicity, and define $R(p) = p \wedge K(q_1^{FB})$ for $p < rq_1^{FB}$ and $R(p) = p \wedge K(q_1^{FB}) - b(q)$ for $p \geq rq_1^{FB}$. Given claim $R(\cdot)$, the firm's payoff rises by $b(q)$ at q_1^{FB}, giving the same incentive as the bonus effect without the distortion away from q_1^{FB}. However $R(\cdot)$ is clearly decreasing at $p = rq_1^{FB}$ and hence not feasible.

Besides, raising capacity to $q > q_1^{FB}$ also causes a deviation $\Delta P_{e,q} - b(q)$ from $b(q)$ that increases across demand realizations in $(q_1^{FB}, q]$, which boosts incentives due to MLRP. This *demand differentiation effect* stems from the assumed non-contractibility of unmet demands. Indeed, rather than

[12] This requirement is standard in and specific to finance models. Without it, there would be no bonus effect. For example, the bonus effect cannot possibly arise in agency models such as Dai and Jerath (2013) as these do not feature a monotonicity condition.

Figure 2 Optimal Capacity q^* as a function of internal funds W



Note. $\Delta\kappa = 13.9$, $r_1 = 11$, $c_1 = 10$, $s_1 = 0$, $D_1 \sim Gamma(10, 10)$ and $D_0 \sim Gamma(8, 10)$.

distorting capacity away from q_1^{FB}, the firm would prefer the financial claim to specify repayments differing across demand realizations in $(q_1^{FB}, q]$. But that is impossible since, for capacity q_1^{FB}, these demand realizations are unmet, and hence non-contractible.

6.3. The Impact of Internal Funds on Investment

The analysis has further implications for internal funds' impact on capacity levels.

Proposition 4. *Two non-negative thresholds $\underline{W}$ and $\overline{W}$ exist such that*
(i) If $W < \underline{W}$, the project is abandoned ($q^* = 0$).
(ii) If $W \in [\underline{W}, \overline{W})$, capacity exceeds the first best ($q^* > q_1^{FB}$) and decreases with cash W.
(iii) If $W \geq \overline{W}$, the first-best capacity is set up ($q^* = q_1^{FB}$).

For cash low enough, the firm abandons the project as it cannot both raise sufficient funds and maintain incentives. For intermediate cash levels, the firm can do both but at the cost of distorting capacity beyond the efficient level to benefit from sufficiently powerful *bonus* and *demand differentiation* effects. As cash increases within that range, the need for external funds drops, as does the debt's face value which also relaxes the incentive compatibility constraint. As a result, less powerful bonus and demand differentiation effects suffice to maintain incentives and capacity can be lowered. The firm sets capacity at the first-best level only when cash is large enough.

Figure 2 depicts the impact of cash W on optimal capacity q^* for the settings of Figure 1 with $\Delta\kappa = 13.9$. As cash W increases, the firm starts raising funds when $W = \underline{W} = 4.2$, then optimal capacity decreases until $W = \overline{W} = 35.3$, at which point the firm sets $q^* = q_1^{FB} = 61$.

7. Strategic Investor

We have assumed a competitive investor. Yet, financiers can enjoy substantial market power over fund-seeking firms. Hence, in this section, we study the opposite case in which the investor is a monopolist. Our over-investment result continues to hold. For brevity, we assume the firm has no cash ($W = 0$). (The results can easily be extended to the case where $W > 0$.)

The game unfolds as follows. In stage 1, the investor offers the firm a contract specifying a capacity $q \geq 0$, a funding amount I, and a feasible financial claim R, which the firm accepts or rejects in stage 2. Rejection implies no investor funding which, given $W = 0$, triggers project abandonment. Instead, acceptance means the contract is implemented: capacity q financed from I is set up; In stage 3, the firm chooses effort $e \in \{0, 1\}$ and incurs cost κ_e; In stage 4, demand D_e is drawn from distribution f_e, revenue $P_{e,q}$ realized and the firm pays $R(P_{e,q})$ to the investor.

7.1. The Investor's Problem

The investor's problem in stage 1 is to choose capacity $q \geq 0$, funding I, and feasible claim $R(\cdot)$ to maximize his expected payoff

$$\max_{q,I,R(\cdot)} \mathbb{E}\left[R(P_{e,q})\right] - I \tag{28}$$

subject to the investor's participation constraint

$$\mathbb{E}\left[R\left(P_{e,q}\right)\right] \geq I \tag{29}$$

the firm's participation constraint

$$\text{if } q > 0, \quad \mathbb{E}\left[P_{e,q} - R\left(P_{e,q}\right)\right] + I - cq - \kappa_e \geq 0 \tag{30}$$

the funding constraint

$$I \geq cq \tag{31}$$

and the incentive compatibility constraint,

$$\text{if } q > 0, \quad \mathbb{E}\left[P_{e,q} - R\left(P_{e,q}\right)\right] + I - cq - \kappa_e \geq \mathbb{E}\left[P_{(1-e),q} - R\left(P_{(1-e),q}\right)\right] + I - cq - \kappa_{(1-e)} \tag{32}$$

7.2. Frictionless Financing

As a benchmark, we study the case in which effort is contractible. The problem is as before except for incentive compatibility constraint (32) which no longer applies. We show that despite the investor's monopoly position, efficiency prevails and that the means of financing is irrelevant.

PROPOSITION 5. *If effort is contractible, the first-best outcome obtains ($q^* = q_1^{FB}$ and $e^* = 1$). The firm and the investor's payoffs are respectively zero and $\pi_1(q_1^{FB}) - \kappa_1$. Any financial claim $R(\cdot)$ is optimal provided $\mathbb{E}\left[R\left(P_{1,q_1^{FB}}\right)\right] \geq cq_1^{FB} + (\pi_1(q_1^{FB}) - \kappa_1)$.*

This result confirms that in an optimal contracting approach, monopoly power *per se* is no financial friction and is thus irrelevant for financing and investment decisions. Indeed, once funding is contractible, holding funding I constant, the efficient outcome obtains with *any* financial claims (debt, equity, convertible debt, warrants, etc.) with the same expected payoff, i.e., $\mathbb{E}[R(P_{1,q_1^{FB}})]$. The efficiency result derives from the Coase Theorem (Coase 1960, Posner 1993) stating that frictionless negotiation by strategic agents leads to the efficient outcome via compensating transfers among agents. Here, transfers take the form of funding I and the expectation of repayments, $\mathbb{E}[R(P_{1,q})]$. Hence, claim R's only relevant aspect is this expectation, its actual shape being irrelevant.

7.3. Debt's optimality and Investment Decision

We now return to the non-contractible effort case. First, the project being viable only for effort $e = 1$, the investor's optimal contract must induce the firm to work, or lead to project abandonment.

LEMMA 3. *Either the firm works ($e = 1$) or the project is abandoned ($q = 0$).*

Given this, the investor's problem can be written as follows:

$$\max_{q,I,R(\cdot)} \mathbb{E}[R(P_{1,q})] - I \tag{33}$$

$$s.t.$$

$$\mathbb{E}[R(P_{1,q})] \geq I \tag{34}$$

$$\mathbb{E}[P_{1,q} - R(P_{1,q})] + I - cq - \kappa_1 \geq 0 \tag{35}$$

$$I \geq cq \tag{36}$$

$$\mathbb{E}[P_{1,q} - R(P_{1,q})] - \mathbb{E}[P_{0,q} - R(P_{0,q})] - \Delta\kappa \geq 0 \tag{37}$$

and if the previous problem is not feasible then the firm abandons the project ($q = 0$).

Recall that to satisfy our model's assumption, the effort costs (κ_0 and κ_1) must be such that $\pi_0(q_0^{FB}) < \kappa_0 < \kappa_1 < \pi_1(q_1^{FB})$ or equivalently $\pi_0(q_0^{FB}) < \kappa_0$ and $0 < \Delta\kappa < \Delta\kappa^{\max}$ with $\Delta\kappa^{\max} \equiv \pi_1(q_1^{FB}) - \kappa_0$. We show that for a given capacity, debt financing is optimal and characterize the optimal debt claim.

Proposition 6. *For all $\Delta\kappa$, a unique threshold $\hat{q} < q_1^{FB}$ exists such that:*

- *No capacity $q \in (0, \hat{q})$ can be optimal.*
- *For any $q \in [\hat{q}, \bar{q}_1]$, a unique debt claim with face value $\bar{K}(q) \in [sq, rq]$ exists such that incentive compatibility constraint (37) is binding, i.e., the unique solution to*

$$(r - s)\int_{(K-sq)/(r-s)}^{q} \left(\bar{F}_1 - \bar{F}_0\right)(u)\,du - \Delta\kappa = 0 \tag{38}$$

Moreover, financing capacity q with that debt claim is (weakly) optimal.

As an important difference with the competitive investor case, and thus with Innes (1990), an optimal debt claim saturates the incentive compatibility constraint, not the funding constraint. For a given capacity q, define the firm's minimum rent, i.e., the smallest expected payoff needed to induce effort. It is condition (37)'s LHS under debt financing with face value $\bar{K}(q)$, i.e.,

$$\Gamma(q) \equiv \mathbb{E}\left[P_{1,q} - \bar{K}(q) \wedge P_{1,q}\right] - (\kappa_0 + \Delta\kappa) \tag{39}$$

The investor's expected payoff can be expressed as total surplus, i.e., the project's value $\pi_1(q) - \kappa_1$, net of the investor's rent $\Gamma(q)$ and of the transfer in excess of funding needs $I - cq$, if any. The investor's problem can thus be rewritten as

$$\max_{q\in[\hat{q},\bar{q}_1], I\geq 0} (\pi_1(q) - \kappa_1) - \Gamma(q) - (I - cq) \tag{40}$$

$$s.t.$$

$$(\pi_1(q) - \kappa_1) - \Gamma(q) - (I - cq) \geq 0 \tag{41}$$

$$\Gamma(q) + (I - cq) \geq 0 \tag{42}$$

$$I - cq \geq 0 \tag{43}$$

$$R(\cdot) \text{ is the debt claim with face value } \bar{K}(q) \tag{44}$$

Since the objective decreases with I which is only bounded from below by constraints (42) and (43), one of them is binding. The problem can thus be rewritten as

$$\max_{q\in[\hat{q},\bar{q}_1]} (\pi_1(q) - \kappa_1) - [\Gamma(q)]^+ \tag{45}$$

$$s.t.$$

$$(\pi_1(q) - \kappa_1) - [\Gamma(q)]^+ \geq 0 \tag{46}$$

$$I = cq - (\Gamma(q))^- \quad \text{and} \quad R \text{ is the debt claim with face value } \bar{K}(q) \tag{47}$$

and a solution exists if and only if

$$\max_{q\in[\hat{q},\bar{q}_1]} (\pi_1(q) - \kappa_1) - [\Gamma(q)]^+ \geq 0 \tag{48}$$

in which case the solution is

$$q^* = \arg\max_{q \in [\hat{q}, \bar{q}_1]} (\pi_1(q) - \kappa_1) - (\Gamma(q))^+ \tag{49}$$

and otherwise the project is abandoned ($q^* = 0$).

THEOREM 2. *Given κ_0, two thresholds $\Delta\underline{\kappa}$ and $\Delta\bar{\kappa}$ exist with $0 < \Delta\underline{\kappa} \leq \Delta\bar{\kappa} \leq \Delta\kappa^{\max}$ such that:*

- *For $\Delta\kappa < \Delta\underline{\kappa}$, the first-best obtains: $q^* = q_1^{FB}$ and $e^* = 1$.*
- *For $\kappa_1 \in (\Delta\underline{\kappa}, \Delta\bar{\kappa}]$, the optimal capacity exceeds the first-best level: $q^* > q_1^{FB}$ and $e^* = 1$.*
- *For $\kappa_1 > \Delta\bar{\kappa}$, the project is abandoned: $q^* = 0$.*

The intuition is as follows. For $\Delta\kappa$ low enough, the rent $\Gamma(q_1^{FB})$ needed to induce the firm to exert $e = 1$ for $q = q_1^{FB}$ is below its reservation utility. Hence, there is no cost for the investor to leave the rent to the firm and "top it up" with a transfer $I > cq_1^{FB}$ to ensure the firm's participation.

For larger values of $\Delta\kappa$, the rent $\Gamma(q_1^{FB})$ exceeds the firm's reservation utility and is therefore costly for the investor. The investor can cut the rent by increasing capacity above the first best level. The cost of doing so it to lower the total surplus generated by the project. However at $q = q_1^{FB}$, that cost is second order and it is therefore optimal for the investor to increase capacity.

For $\Delta\kappa$ large enough, for any capacity level, the rent exceeds the project's value. Hence inducing effort is incompatible with the investor's participation constraint, and the project is abandoned.

8. Conclusion

This paper follows an optimal contracting approach to study how a firm's capacity choice under demand uncertainty interacts with its decision of how to finance that capacity. The question is relevant only under financial frictions, which we model as a moral hazard problem.

In the optimal contracting approach, the firm optimizes over a set of available sources of funds derived endogenously from assumptions about fundamentals, i.e., preferences, physical constraints, and contractibility. This is in sharp contrast to the existing literature on financing capacity, which assumes exogenously restricted sets of feasible claims. These restrictions sometimes lead one to conclude capacity should be distorted away from first best, when in fact that need would disappear at no cost with a simple contract change. Ours is the first paper to propose results for the problem of financing capacity under demand uncertainty, which are robust in this sense.

Specifically, given optimal financing, we characterize the situations where the optimal capacity is at the efficient level and when it is *strictly above* that level. We also determine when the firm cannot fund the project and the capacity level is set at zero, below the efficient level. Further, provided the firm can finance the project, we show that the optimal capacity level is non-decreasing in the moral hazard problem's severity and non-increasing in the firm's internal funds.

That the optimal capacity can exceed the efficient level runs counter to the common intuition that by raising the unit capacity cost, financial market frictions lead to lower capacity investment. We trace the value of increasing capacity beyond the efficient level to a *bonus effect* and a *demand differentiation effect*. Both effects relate to financial claim feasibility constraints, which are standard in finance, and the risk of unmet demands, which is characteristic of capacity choices under demand uncertainty.

Our work suggests that, before performing costly deviations from efficient capacity investment decisions, operations managers may want to challenge the financial policy and, more generally, check whether adjustments to financial contracts can avoid these distortions. Indeed, adjusting a contract is likely cheaper than deviating from the optimal capacity level. When no adjustment in the contract can fully eliminate the problem, over-investment, rather than under-investment, may help the manager mitigate financial frictions.

Our model can be extended in many directions. In particular, while effort affects demand in our model, other types of moral hazard are possible. For instance, effort may affect the quality or yield of installed capacity. In this case, we suspect that the optimal capacity is typically below the efficient level. Indeed, this form of financial frictions should increase with the level of investment and thus incur higher costs of external capital for higher levels of capacity. This is in contrast with the bonus and demand differentiation effects we identify in this paper, which cause the frictions to become less severe for higher capacity.

Further, we have focused on the natural assumption that higher demand realizations are more indicative of high effort. This need not be the case. For instance, effort could affect (reduce) the variance of demand. In this case, MLRP would not hold and outside equity could provide a better source of funding than debt. We also focus on financial frictions stemming from moral hazard. Other types of frictions, however, are possible such as asymmetric information.

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Appendix A: Preliminary Lemmas

LEMMA 4. *For all $q \in \mathbb{R}_+$, $g_{1,q}/g_{0,q}$ is strictly increasing over $[sq, rq]$, where $g_{e,q}$ is the probability distribution of P_e, q.*

Proof: The distributions of payoff are$g_{e,q}(x) = f_e\left(P_{e,q}^{-1}(x)\right)$ for $x \in [sq, rq)$ and $g_{e,q}(rq) = \bar{F}_e(q)$. Hence $g_{1,q}/g_{0,q}$ is increasing over $[sq, rq)$ because f_0 and f_1 satisfy MLRP. Moreover, we have

$$\begin{aligned}
\frac{g_{1,q}(rq)}{g_{0,q}(rq)} &= \frac{\bar{F}_1(q)}{\bar{F}_0(q)} = \frac{\int_q^{+\infty} f_1(x)\,dx}{\int_q^{+\infty} f_0(x)\,dx} = \frac{\int_q^{+\infty} \frac{f_1(x)}{f_0(x)} f_0(x)\,dx}{\int_q^{+\infty} f_0(x)\,dx} > \frac{\int_q^{+\infty} \frac{f_1(q)}{f_0(q)} f_0(x)\,dx}{\int_q^{+\infty} f_0(x)\,dx} \text{ by MLRP} \\
&> \frac{f_1(q)}{f_0(q)} > \frac{f_1(x)}{f_0(x)} \qquad \text{for all } x \in [0, q) \text{ by MLRP} \\
&> \frac{g_{1,q}(p)}{g_{0,q}(p)} \qquad \text{for all } p \in [sq, rq)
\end{aligned}$$

■

LEMMA 5. *For $q \in \mathbb{R}_+$, consider two feasible claims A and B with repayment functions $R_A(\cdot)$ and $R_B(\cdot)$ satisfying $\mathbb{E}[R_A(P_{1,q})] = \mathbb{E}[R_B(P_{1,q})]$ and $\exists p^* \in (sq, rq)$ such that $\forall p \in [sq, p^*)$ $R_A(p) \geq R_B(p)$ and $\forall p \in (p^*, rq]$ $R_A(p) \leq R_B(p)$ with strict inequalities for a set of payoffs. Constraint (19)'s left-hand side is strictly larger for A than for B.*

Proof: We want to show that the left-hand side of (19) is weakly larger for A than for B, i.e.,

$$\mathbb{E}[P_{1,q} - R_A(P_{1,q})] - \mathbb{E}[P_{0,q} - R_A(P_{0,q})] - \Delta\kappa > \mathbb{E}[P_{1,q} - R_B(P_{1,q})] - \mathbb{E}[P_{0,q} - R_B(P_{0,q})] - \Delta\kappa.$$

After simplification and using condition $\mathbb{E}[R_A(P_{1,q})] = \mathbb{E}[R_B(P_{1,q})]$, this amounts to showing that $\mathbb{E}[\Delta(P_{0,q})] > 0$, where $\Delta(\cdot) \equiv R_A(\cdot) - R_B(\cdot)$. Therefore, we have

$$\begin{aligned}
\mathbb{E}[\Delta(P_{0,q})] &= \int_{sq}^{p^*} \Delta(p)\, g_{0,q}(p)\,dp - \int_{p^*}^{rq} -\Delta(p)\, g_{0,q}(p)\,dp = \int_{sq}^{p^*} \Delta(p) \frac{g_{0,q}(p)}{g_{1,q}(p)} g_{1,q}(p)\,dp - \int_{p^*}^{rq} -\Delta(p) \frac{g_{0,q}(p)}{g_{1,q}(p)} g_{1,q}(p)\,dp \\
&> \frac{g_{0,q}(p^*)}{g_{1,q}(p^*)} \int_{sq}^{p^*} \Delta(p)\, g_{1,q}(p)\,dp - \frac{g_{0,q}(p^*)}{g_{1,q}(p^*)} \int_{p^*}^{rq} -\Delta(p)\, g_{1,q}(p)\,dp > \frac{g_{0,q}(p^*)}{g_{1,q}(p^*)} \mathbb{E}[\Delta(P_{1,q})] > 0
\end{aligned}$$

LEMMA 6. *If $W \geq (c-s)q$ then $\partial \hat{d}(q)/\partial q = 0$ and $\partial K(q)/\partial q = c$. If $W < (c-s)q$ then*

$$\frac{\partial \hat{d}(q)}{\partial q} = \frac{\bar{F}_1(q_1^{FB})}{\bar{F}_1\left(\hat{d}(q)\right)} \quad \text{and} \quad \frac{\partial K(q)}{\partial q} = \frac{c-s}{\bar{F}_1\left(\hat{d}(q)\right)} + s.$$

Proof: If $W \geq (c-s)q$, (22) implies $K(q) \leq sq$ and $\hat{d}(q) = 0$. Therefore $\partial \hat{d}(q)/\partial q = 0$, and taking the first-order derivative of condition (22) with respect to q we deduce yields $\partial K/(q)\,\partial q = c$.

If $W < (c-s)q$, (22) implies $K(q) > sq$ and $\hat{d}(q) = (K(q) - sq)/(r-s)$. Hence,

$$\frac{\partial K(q)}{\partial q} - (r-s)\frac{\partial \hat{d}(q)}{\partial q} F_1\left(\frac{K - sq}{r-s}\right) = c \tag{50}$$

Since $\partial \hat{d}(q)/\partial q = (\partial K(q)/\partial q - s)/(r-s)$ this expression can be rewritten as

$$s + (r-s)\frac{\partial \hat{d}(q)}{\partial q} - (r-s)\frac{\partial \hat{d}(q)}{\partial q} F_1\left(\hat{d}(q)\right) = c$$

$$\text{or,} \quad \frac{\partial \hat{d}(q)}{\partial q} = \frac{c-s}{r-s} \frac{1}{1 - F_1\left(\hat{d}(q)\right)} = \left(1 - \frac{r-c}{r-s}\right) \frac{1}{1 - F_1\left(\hat{d}(q)\right)} = \frac{\bar{F}_1(q_1^{FB})}{\bar{F}_1\left(\hat{d}(q)\right)}$$

and the corresponding derivative of $K(q)$ is obtained from (50). ■

Appendix B: Proofs

B.1. Proof of Proposition 2

Step 1. We adapt Innes (1990) to show we can focus on debt claims. For a given capacity $q \in (0, \bar{q}_1]$, the firm's problem is to find a claim $R(\cdot)$ satisfying (17), (18) and (19). Consider such a claim, $R(\cdot)$, assuming one exists, and consider the debt claim with face value $K \in (sq, rq)$ defined as the unique solution to

$$\mathbb{E}\left[P_{1,q} \wedge K\right] = \mathbb{E}\left[R\left(P_{1,q}\right)\right] \tag{51}$$

Note that by construction, the debt claim satisfies (17), because contract $R(\cdot)$ does. Also, like all debt claims, it satisfies (18).

Notice that (18) implies that $\exists p^* \in (sq, rq)$ such that $\forall p \leq p^*$, $R(p) \leq p \wedge K$ and $\forall p \geq p^*$, $R(p) \geq p \wedge K$. Hence, by Lemma 5, (19) is strictly tighter for contract $R(\cdot)$ than for the debt claim. In that sense, $R(\cdot)$ is dominated by the debt claim.

Step 2. Define function h as

$$h(K) \equiv K - (r-s)\int_0^{\left(\frac{K-sq}{r-s}\right)^+} F_1(x)\,dx - (cq-W)^+$$

Expression (22) is written as $h(K) = 0$. Note that $h(K)$ is strictly increasing in K as $h'(K) = 1$ for $K \leq sq$ and $h'(K) = 1 - F_1\left(\frac{K-sq}{r-s}\right)$ for $K \geq sq$.

If $W \geq (c-s)q$ then $(cq-W)^+ \leq sq$ and $h\left((cq-W)^+\right) = (cq-W)^+ - (r-s)\int_0^0 F_1(x)\,dx - (cq-W)^+ = 0$. Therefore $K(q) = (cq-W)^+ \in [0, sq]$. Debt is riskfree because $K(q) \leq sq$ while $P_{1,q} \geq sq$. Conversely, if $W < (c-s)q$ then $(cq-W)^+ > sq$ and so $h(sq) = sq - (r-s)\int_0^0 F_1(x)\,dx - (cq-W)^+ < 0$. Moreover

$$h(rq) = rq - (r-s)\int_0^q F_1(x)\,dx - (cq-W)^+ = \left[\pi_1(q) - \kappa_1\right] + \left[\kappa(1) + cq - (cq-W)^+\right] > 0.$$

Indeed, the first bracket term non-negative over $[0, \bar{q}_1]$ and the second one is strictly positive because $\kappa_1 > 0$.

Therefore $K(q) \in (sq, rq)$. Moreover, debt is risky because $K(q) > sq$ and $P_{1,q} \geq sq$.

B.2. Proof of Lemma 2

We have

$$\frac{\partial L(q,\Delta\kappa)}{\partial q} = \left(F_0(q) - F_1(q)\right) - \frac{\partial \hat{d}(q)}{\partial q}\left(F_0\left(\hat{d}(q)\right) - F_1\left(\hat{d}(q)\right)\right)$$

If $W \geq (c-s)q$, $\hat{d}(q) = 0$ from Lemma 6 (see Appendix A), which implies $\partial L(q,\Delta\kappa)/\partial q = (F_0(q) - F_1(q)) > 0$. This means that $L(q,\Delta\kappa)$ is increasing in q and that $q_1^{\max} = \bar{q}_1 > q_1^{FB}$.

If $W < (c-s)q$, $\partial \hat{d}(q)/\partial q = \bar{F}_1(q_1^{FB})/\bar{F}_1\left(\hat{d}(q)\right)$ from Lemma 6. Hence, for $W < (c-s)q_1^{FB}$, we can rewrite (B.2) as

$$\begin{aligned}\frac{\partial L(q,\Delta\kappa)}{\partial q} &= \left(F_0(q) - F_1(q)\right) - \frac{\bar{F}_1(q_1^{FB})}{\bar{F}_1\left(\hat{d}(q)\right)}\left(F_0\left(\hat{d}(q)\right) - F_1\left(\hat{d}(q)\right)\right) \\ &= \left(\bar{F}_1(q) - \bar{F}_0(q)\right) - \frac{\bar{F}_1(q_1^{FB})}{\bar{F}_1\left(\hat{d}(q)\right)}\left(\bar{F}_1\left(\hat{d}(q)\right) - \bar{F}_0\left(\hat{d}(q)\right)\right)\end{aligned}$$

For $q \le q_1^{FB}$, we have $\bar{F}_1(q) \ge \bar{F}_1(q_1^{FB})$ and so

$$\begin{aligned}
\frac{\partial L(q,\Delta\kappa)}{\partial q} &\ge \left(\bar{F}_1(q) - \bar{F}_0(q)\right) - \frac{\bar{F}_1(q)}{\bar{F}_1\left(\hat{d}(q)\right)}\left(\bar{F}_1\left(\hat{d}(q)\right) - \bar{F}_0\left(\hat{d}(q)\right)\right) \\
&\ge \bar{F}_1(q)\left[\left(1 - \frac{\bar{F}_0(q)}{\bar{F}_1(q)}\right) - \left(1 - \frac{\bar{F}_0\left(\hat{d}(q)\right)}{\bar{F}_1\left(\hat{d}(q)\right)}\right)\right] = \bar{F}_1(q)\left[\frac{\bar{F}_0\left(\hat{d}(q)\right)}{\bar{F}_1\left(\hat{d}(q)\right)} - \frac{\bar{F}_0(q)}{\bar{F}_1(q)}\right]
\end{aligned}$$

Note now that $\bar{F}_0/\bar{F}_1$ is decreasing. Indeed,

$$\frac{d}{dx}\frac{\bar{F}_0(x)}{\bar{F}_1(x)} = \frac{-f_0(x)\bar{F}_1(x) + f_1(x)\bar{F}_0(x)}{\left(\bar{F}_1(x)\right)^2} < 0$$

since MLRP implies that D_1 stochastically dominates D_0 according to the hazard rate order.[13]

Note that for all $q < \bar{q}_1$, $\hat{d}(q) < q$. Hence for all $q \le q_1^{FB}$, $\partial L(q,\Delta\kappa)/\partial q > 0$, which implies $q_1^{\max} > q_1^{FB}$.

B.3. Proof of Theorem 1

Recall that (4) implies an upper bound on $\Delta\kappa$, i.e., $\pi_1(q_1^{FB}) - \pi_0(q_0^{FB}) > \Delta\kappa$. If $\pi_1(q_1^{FB}) - \pi_0(q_0^{FB}) \le (r-s)L(q_1^{FB},0)$, then for all values of $\Delta\kappa$ such that (4) holds, we have $L(q_1^{FB},\Delta\kappa) \ge 0$ so that $q^* = q_1^{FB}$ for all W and we have thresholds $\Delta\underline{\kappa} = \Delta\bar{\kappa} = 0$. Assume now that $\pi_1(q_1^{FB}) - \pi_0(q_0^{FB}) < (r-s)L(q_1^{FB},0)$. If $W \ge (c-s)q_1^{FB}$ then $\hat{d}(q_1^{FB}) = 0$. In that case, (4) implies $L(q_1^{FB},\Delta\kappa) > 0$ and so $q^* = \inf S(\Delta\kappa) = q_1^{FB}$. For all q, $L(q,0) > 0$ and for $\Delta\kappa$ large enough $L(q,\Delta\kappa) < 0$. Moreover, $L(q,\Delta\kappa)$ is continuous and strictly decreasing with $\Delta\kappa$. Therefore, we can define $\Delta\underline{\kappa}$ and $\Delta\bar{\kappa}$ by $L(q_1^{FB},\Delta\underline{\kappa}) = 0$ and $L(\hat{q}_1,\Delta\bar{\kappa}) = 0$.
For $\Delta\kappa \le \Delta\underline{\kappa}$, $L(q_1^{FB},\Delta\kappa) \ge 0$ and so $q^* = \inf S(\Delta\kappa) = q_1^{FB}$. For $\Delta\kappa > \Delta\bar{\kappa}$, $S(\Delta\kappa) = \varnothing$ and $q^* = 0$.
For $\Delta\underline{\kappa} < \Delta\kappa \le \Delta\bar{\kappa}$, $L(q_1^{FB},\Delta\kappa) < 0$ and $L(\hat{q}_1,\Delta\kappa) \ge 0$ so $q^* = \inf S(\Delta\kappa) > q_1^{FB}$ and increases with $\Delta\kappa$.

B.4. Proof of Proposition 3

We have

$$\begin{aligned}
\mathbb{E}[\Delta P_{e,q}] &= \bar{F}_e\left(q_1^{FB}\right)\cdot\mathbb{E}\left[\Delta P_{e,q}|D_e \ge q_1^{FB}\right] + F_e\left(q_1^{FB}\right)\cdot\mathbb{E}\left[\Delta P_{e,q}|D_e \le q_1^{FB}\right] \\
&= \bar{F}_e\left(q_1^{FB}\right)\left(b(q) + a_e(q)\right) + F_e\left(q_1^{FB}\right)\cdot\mathbb{E}\left[\Delta P_{e,q}|D_e \le q_1^{FB}\right]
\end{aligned}$$

Hence given $(r-s)L(q,\Delta\kappa) = (\mathbb{E}[P_{1,q}] - \mathbb{E}[P_{0,q}]) - (\mathbb{E}[K(q)\wedge P_{1,q}] - \mathbb{E}[K(q)\wedge P_{0,q}]) - \Delta\kappa$, we have

$$\begin{aligned}
(r-s)\left(L(q,\Delta\kappa) - L\left(q_1^{FB},\Delta\kappa\right)\right) &= (\mathbb{E}[\Delta P_{1,q}] - \mathbb{E}[\Delta P_{0,q}]) - (\mathbb{E}[K(q)\wedge P_{1,q}] - \mathbb{E}[K(q)\wedge P_{0,q}]) \\
&\quad + \left(\mathbb{E}\left[K\left(q_1^{FB}\right)\wedge P_{1,q_1^{FB}}\right] - \mathbb{E}\left[K\left(q_1^{FB}\right)\wedge P_{0,q_1^{FB}}\right]\right) \\
&= \begin{pmatrix} \bar{F}_1(q_1^{FB})\cdot\mathbb{E}[\Delta P_{1,q}|D_1 \ge q_1^{FB}] \\ +F_1(q_1^{FB})\cdot\mathbb{E}[\Delta P_{1,q}|D_1 \le q_1^{FB}] \end{pmatrix} - \begin{pmatrix} \bar{F}_0(q_1^{FB})\cdot\mathbb{E}[\Delta P_{1,q}|D_0 \ge q_1^{FB}] \\ +F_0(q_1^{FB})\cdot\mathbb{E}[\Delta P_{0,q}|D_0 \le q_1^{FB}] \end{pmatrix} \\
&\quad - (\mathbb{E}[K(q)\wedge P_{1,q}] - \mathbb{E}[K(q)\wedge P_{0,q}]) + \mathbb{E}\left[K\left(q_1^{FB}\right)\wedge P_{1,q_1^{FB}}\right] - \mathbb{E}\left[K\left(q_1^{FB}\right)\wedge P_{0,q_1^{FB}}\right] \\
&= \beta(q) + \alpha(q) - \varphi(q)
\end{aligned}$$

where denoting $\Delta\bar{F} \equiv \bar{F}_1 - \bar{F}_0$ and $\Delta f \equiv f_1 - f_0$ we have

$$\varphi(q) \equiv F_0\left(q_1^{FB}\right)\cdot\mathbb{E}\left[\Delta P_{0,q}|D_0 \le q_1^{FB}\right] - F_1\left(q_1^{FB}\right)\cdot\mathbb{E}\left[\Delta P_{1,q}|D_1 \le q_1^{FB}\right]$$

[13] To see this, note that $f_1(x)\bar{F}_0(x) = f_0(x)\int_x^{+\infty}(f_1(x)/f_0(x))/(f_1(t)/f_0(t))f_1(t)\,dt \le f_0(x)\int_x^{+\infty}f_1(t)\,dt = f_0(x)\bar{F}_1(x)$ where the inequality holds since under MLRP, for all $t > x$, we have $f_1(x)/f_0(x) < f_1(t)/f_0(t)$.

$$
\begin{aligned}
&+\left(\mathbb{E}\left[K(q)\wedge P_{1,q}\right]-\mathbb{E}\left[K(q)\wedge P_{0,q}\right]\right)-\left(\mathbb{E}\left[K\left(q_1^{FB}\right)\wedge P_{1,q_1^{FB}}\right]-\mathbb{E}\left[K\left(q_1^{FB}\right)\wedge P_{0,q_1^{FB}}\right]\right)\\
&=(r-s)\int_0^{\hat{d}(q)}\Delta\bar{F}(x)\,dx-\int_0^{q_1^{FB}} s\left(q-q_1^{FB}\right)\Delta f(x)\,dx\\
&-\left(\mathbb{E}\left[K\left(q_1^{FB}\right)\wedge P_{1,q_1^{FB}}\right]-\mathbb{E}\left[K\left(q_1^{FB}\right)\wedge P_{0,q_1^{FB}}\right]\right)
\end{aligned}
$$

Hence

$$
\frac{\partial L(q,\Delta\kappa)}{\partial q}=\left(\frac{\partial\beta(q)}{\partial q}+\frac{\partial\alpha(q)}{\partial q}-\frac{\partial\varphi(q)}{\partial q}\right)/(r-s)
$$

Since $\partial\hat{d}(q)/\partial q>0$, we have

$$
\frac{\partial\varphi(q)}{\partial q}=(r-s)\left(\frac{\partial\hat{d}(q)}{\partial q}\right)\Delta\bar{F}\left(\hat{d}(q)\right)+s\Delta\bar{F}\left(q_1^{FB}\right)>0
$$

Besides, we have

$$
\begin{aligned}
\beta(q)&=\left(\int_{q_1^{FB}}^{q}(sq+(r-s)x)f_1(x)\,dx+rq\bar{F}_1(q)-rq_1^{FB}\bar{F}_1\left(q_1^{FB}\right)\right)\Delta\bar{F}\left(q_1^{FB}\right)/\bar{F}_1\left(q_1^{FB}\right)\\
&=\left(sq\left(F_1(q)-F_1\left(q_1^{FB}\right)\right)+\int_{q_1^{FB}}^{q}(r-s)xf_1(x)\,dx+rq\bar{F}_1(q)-rq_1^{FB}\bar{F}_1\left(q_1^{FB}\right)\right)\frac{\Delta\bar{F}\left(q_1^{FB}\right)}{\bar{F}_1\left(q_1^{FB}\right)}\\
\frac{\partial\beta(q)}{\partial q}&=\frac{\Delta\bar{F}\left(q_1^{FB}\right)}{\bar{F}_1\left(q_1^{FB}\right)}\left(s\left(F_1(q)-F_1\left(q_1^{FB}\right)\right)+sqf_1(q)+(r-s)qf_1(q)+r\bar{F}_1(q)-rqf_1(q)\right)\\
&=\left[s\left(F_1(q)-F_1\left(q_1^{FB}\right)\right)+r\bar{F}_1(q)\right]\Delta\bar{F}\left(q_1^{FB}\right)/\bar{F}_1\left(q_1^{FB}\right)>0.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
\alpha(q)&=\mathbb{E}\left[\Delta P_{1,q}-b(q)|D_1\geq q_1^{FB}\right]\bar{F}_1\left(q_1^{FB}\right)-\mathbb{E}\left[\Delta P_{0,q}-b(q)|D_0\geq q_1^{FB}\right]\bar{F}_0\left(q_1^{FB}\right)\\
&=\mathbb{E}\left[\Delta P_{1,q}|D_1\geq q_1^{FB}\right]\bar{F}_1\left(q_1^{FB}\right)-\mathbb{E}\left[\Delta P_{0,q}|D_0\geq q_1^{FB}\right]\bar{F}_0\left(q_1^{FB}\right)-b(q)\Delta\bar{F}\left(q_1^{FB}\right)\\
&=\left(\mathbb{E}\left[P_{1,q}|D_1\geq q_1^{FB}\right]-rq_1^{FB}\right)\bar{F}_1\left(q_1^{FB}\right)-\left(\mathbb{E}\left[P_{0,q}|D_0\geq q_1^{FB}\right]-rq_1^{FB}\right)\bar{F}_0\left(q_1^{FB}\right)-\beta(q)\\
&=\int_{q_1^{FB}}^{q}(sq+(r-s)x)\Delta f(x)\,dx+rq\Delta\bar{F}(q)-rq_1^{FB}\Delta\bar{F}\left(q_1^{FB}\right)-\beta(q)\\
\frac{\partial\alpha(q)}{\partial q}&=s\left(\Delta F(q)-\Delta F\left(q_1^{FB}\right)\right)+rq\Delta f(q)+r\Delta\bar{F}(q)-rq\Delta f(q)-\frac{\partial\beta(q)}{\partial q}\\
&=s\left(\Delta F(q)-\Delta F\left(q_1^{FB}\right)\right)+r\Delta\bar{F}(q)-\frac{1}{\bar{F}_1\left(q_1^{FB}\right)}\left(s\left(F_1(q)-F_1\left(q_1^{FB}\right)\right)+r\bar{F}_1(q)\right)\Delta\bar{F}\left(q_1^{FB}\right)\\
&=(r-s)\bar{F}_1(q)\left(\frac{\bar{F}_0\left(q_1^{FB}\right)}{\bar{F}_1\left(q_1^{FB}\right)}-\frac{\bar{F}_0(q)}{\bar{F}_1(q)}\right)
\end{aligned}
$$

which is zero for $q=q_1^{FB}$ and strictly positive for $q>q_1^{FB}$ due to MLRP. Indeed,

$$
\begin{aligned}
\frac{\partial}{\partial q}\left(\frac{\bar{F}_0(q)}{\bar{F}_1(q)}\right)&=\frac{-f_0(q)\bar{F}_1(q)+f_1(q)\bar{F}_0(q)}{\left(\bar{F}_1(q)\right)^2}=\frac{f_1(q)\bar{F}_1(q)}{\left(\bar{F}_1(q)\right)^2}\left(\frac{\bar{F}_0(q)}{\bar{F}_1(q)}-\frac{f_0(q)}{f_1(q)}\right)\\
&=\frac{f_1(q)\bar{F}_1(q)}{\left(\bar{F}_1(q)\right)^2}\left(\frac{\int_q^{+\infty}f_0(x)\,dx}{\bar{F}_1(q)}-\frac{f_0(q)}{f_1(q)}\right)=\frac{f_1(q)\bar{F}_1(q)}{\left(\bar{F}_1(q)\right)^2}\left(\frac{\int_q^{+\infty}\frac{f_0(x)}{f_1(x)}f_1(x)\,dx}{\bar{F}_1(q)}-\frac{f_0(q)}{f_1(q)}\right)\\
&>\frac{f_1(q)\bar{F}_1(q)}{\left(\bar{F}_1(q)\right)^2}\left(\frac{\int_q^{+\infty}\frac{f_0(q)}{f_1(q)}f_1(x)\,dx}{\bar{F}_1(q)}-\frac{f_0(q)}{f_1(q)}\right)\quad\text{by MLRP}\\
&>0
\end{aligned}
$$

B.5. Proof of Proposition 4

Consider the left-hand side of (25) as a function $L(W,q,\Delta\kappa)$. We have

$$L\left(0,q_1^{FB},0\right)=\int_{\hat{d}\left(q_1^{FB}\right)}^{q_1^{FB}}\left(F_0(x)-F_1(x)\right)dx>0$$

Recall that constraint (4) implies an upper bound on $\Delta\kappa$, i.e., $\pi_1(q_1^{FB})-\pi_0(q_0^{FB})>\Delta\kappa$.

Case 1: $\pi_1(q_1^{FB})-\pi_0(q_0^{FB})<(r-s)L(0,q_1^{FB},0)$. In that case, for all values of $\Delta\kappa$ such that (4) holds we have $L(0,q_1^{FB},\Delta\kappa)>0$. Since $\partial L(W,q,\Delta\kappa)/\partial W\geq 0$, this implies $L(W,q_1^{FB},\Delta\kappa)\geq 0$ and so $q^*=q_1^{FB}$ for all W, i.e. $\underline{W}=\overline{W}=0$.

Case 2: $\pi_1(q_1^{FB})-\pi_0(q_0^{FB})>(r-s)L(0,q_1^{FB},0)$. In that case, values of $\Delta\kappa$ such that condition (4) holds exist such that

$$L\left(0,q_1^{FB},\Delta\kappa\right)=\int_{\hat{d}\left(q_1^{FB}\right)}^{q_1^{FB}}\left(F_0(x)-F_1(x)\right)dx-\frac{\Delta\kappa}{(r-s)}<0$$

Since L is continuous and $\partial L(0,q,\Delta\kappa)/\partial\Delta\kappa<0$, we can define $\Delta\kappa_0$ as the sole solution to $L(0,q_1^{FB},\Delta\kappa)=0$.

For all $\Delta\kappa\leq\Delta\kappa_0$, $L(0,q_1^{FB},\Delta\kappa)\geq 0$ which given $\partial L(W,q,\Delta\kappa)/\partial W\geq 0$ implies $L(W,q_1^{FB},\Delta\kappa)\geq 0$ for all W so that $\underline{W}=\overline{W}=0$. Now consider $\Delta\kappa>\Delta\kappa_0$, which implies $L(0,q_1^{FB},\Delta\kappa)<0$. Note that for $W\geq(c-s)q_1^{FB}$ we have $\hat{d}(q_1^{FB})=0$ which implies

$$\begin{aligned}L\left((c-s)q_1^{FB},q_1^{FB},\Delta\kappa\right) &= \int_0^{q_1^{FB}}\left(F_0(x)-F_1(x)\right)dx-\frac{\Delta\kappa}{(r-s)}\\ &= \frac{\pi_1(q_1^{FB})-\pi_0(q_1^{FB})-\Delta\kappa}{(r-s)}\\ &> \frac{\pi_1(q_1^{FB})-\pi_0(q_0^{FB})-\Delta\kappa}{(r-s)}>0 \text{ due to (4).}\end{aligned}$$

Note that $q_1^{\max}(W)$ is a function of W. If $L(0,q_1^{\max}(0),\Delta\kappa)\geq 0$ then let $\underline{W}=0$. Else define $\underline{W}$ as the smallest solution to $L(W,q_1^{\max}(W),\Delta\kappa)=0$. We have for all $W<\underline{W}$ (if any), $L(W,q_1^{\max}(W),\Delta\kappa)<0$ and so $L(W,q,\Delta\kappa)<0$ for all $q\leq\bar{q}_1$. Hence the project is abandoned. Define $\bar{W}$ as the unique solution to $L(W,q_1^{FB},\Delta\kappa)=0$. Since $L(W,q_1^{FB},\Delta\kappa)$ is non-decreasing in W, we have for all $W\geq\bar{W}$, $L(W,q_1^{FB},\Delta\kappa)\geq 0$ and so $q^*=q_1^{FB}$. For $W=\underline{W}$, $q^*=q_1^{\max}(\underline{W})$ and for all $W\in[\underline{W},\bar{W})$

$$q^*(W)=\inf\left\{q\in\left[q_1^{FB},q_1^{\max}(W)\right]\text{ s.t. }L(W,q,\Delta\kappa)\geq 0\right\}$$

By definition, $q_1^{\max}(W)\leq\bar{q}_1$ for all W. Hence, the previous condition can be rewritten as

$$q^*(W)=\inf\left\{q\in\left[q_1^{FB},\bar{q}_1\right]\text{ s.t. }L(W,q,\Delta\kappa)\geq 0\right\}$$

Since for all q, $\partial L(W,q,\Delta\kappa)/\partial W\geq 0$, $q^*(W)$ decreases with W over $[\underline{W},\bar{W})$.

B.6. Proof of Proposition 5

Effort being contractible, (32) is irrelevant. Objective $\mathbb{E}[R(P_{e,q})]-I$ is bounded above by (30), which must thus be binding. Given this, (28) is $\mathbb{E}[P_{e,q}]-cq-\kappa_e$ which is maximized for $q=q_1^{FB}$ and $e=1$. Finally, the investor and the firm's expected payoffs only depend on $\mathbb{E}[R(P_{1,q})]-I$, not on the shape of $R(\cdot)$.

B.7. Proof of Lemma 3

Suppose $q > 0$. Given (29), (30) implies $\pi_e(q) - \kappa_e \geq 0$ which is violated for $e = 0$ given that $\pi_0(q_0^{FB}) - \kappa_0 < 0$.

B.8. Proof of Proposition 6

First, we prove the optimality of debt financing for all q. In (33), claim $R(\cdot)$ appears only as $\mathbb{E}[R(P_{1,q})]$ except in (37). From Lemma 5, we know that given capacity q, for all claims R, a debt claim with face value K such that with $\mathbb{E}[K \wedge P_{1,q}] = \mathbb{E}[R(P_{1,q})]$ exists, is unique and relaxes (37). Hence given (q, I) that debt claim weakly dominates R.

Next, we show the existence of $\hat{q} < q_1^{FB}$ and the existence and uniqueness of $\bar{K}(q)$ for all $q \geq \hat{q}$. For all $q > 0$ and a given debt claim with face value K, (37)'s LHS can be written as

$$\begin{aligned}
L(q,K) &= \int_{\bar{d}(K)}^{q} (sq + (r-s)u - K)(f_1 - f_0)(u)\,du + (rq - K)\left(\bar{F}_1 - \bar{F}_0\right)(q) - \Delta\kappa \\
&= (rq - K)(F_1 - F_0)(q) - (r-s)\int_{\bar{d}(K)}^{q} (F_1 - F_0)(u)\,du + (rq - K)\left(\bar{F}_1 - \bar{F}_0\right)(q) - \Delta\kappa \\
&= (r-s)\int_{\bar{d}(K)}^{q} \left(\bar{F}_1 - \bar{F}_0\right)(u)\,du - \Delta\kappa
\end{aligned}$$

where $\bar{d}(K) \equiv (K - sq)^+ / (r-s)$. We have $L(0,0) = -\Delta\kappa < 0$ and

$$\begin{aligned}
\frac{\partial L(q,0)}{\partial q} &= \frac{\partial}{\partial q}\left((r-s)\int_0^q \left(\bar{F}_1 - \bar{F}_0\right)(u)\,du - \Delta\kappa\right)(r-s)\left(\bar{F}_1 - \bar{F}_0\right)(q) > 0 \\
\text{Hence,}\quad L\left(q_1^{FB}, 0\right) &= \left(\mathbb{E}\left[P_{1,q_1^{FB}}\right] - cq_1^{FB} - \kappa_1\right) - \left(\mathbb{E}\left[P_{0,q_1^{FB}}\right] - cq_1^{FB} - \kappa_0\right) \\
&> \left(\pi_1\left(q_1^{FB}\right) - \kappa_1\right) - \left(\pi_0\left(q_0^{FB}\right) - \kappa_0\right) > 0
\end{aligned}$$

because by assumption, the first term in brackets is positive and the second one is negative. Hence we can define a unique threshold $\hat{q} < q_1^{FB}$ such that $L(\hat{q}, 0) = 0$. Note also that for all $q > 0$

$$\frac{\partial L(q,K)}{\partial K} = -(r-s)\frac{\partial \bar{d}(K)}{\partial K}\left(\bar{F}_1 - \bar{F}_0\right)\left(\bar{d}(K)\right)$$

Given that $\bar{d}(K) \equiv (K - sq)^+ / (r-s)$, we have

$$\frac{\partial L(q,K)}{\partial K} = 0 \text{ for } K \in [0, sq] \text{ and } \frac{\partial L(q,K)}{\partial K} < 0 \text{ for } K \in [sq, rq]$$

Hence, for all $q \in (0, \hat{q})$ and all $K \geq 0$, we have $L(q,K) \leq L(q,0) < 0$ and (37) is violated. Hence $q \in (0, \hat{q})$ cannot be optimal and we can focus on $q \in [\hat{q}, \bar{q}_1]$. For all $q \in [\hat{q}, \bar{q}_1]$, given that $\bar{d}(rq) = q$, we have $L(q, rq) = 0 - 0 - \Delta\kappa < 0 < L(q, 0)$. Finally, for a given $q \geq \hat{q}$, $L(q,K)$ is non-negative and constant in K over $[0, sq]$ and strictly decreasing in K over $[sq, rq]$. Hence, a unique K exists such that $L(q,K) = 0$. Note that $K \in (sq, rq)$. This proves the existence and uniqueness of $\bar{K}(q)$ for all $q \geq \hat{q}$.

Last, we show the optimality of $\bar{K}(q)$ for a given $q \geq \hat{q}$. We can minimize $\mathbb{E}[R(P_{1,q})]$ so that (37) is binding. Indeed, the objective and all conditions other than (37) depend on $R(\cdot)$ only via $\mathbb{E}[R(P_{1,q})] - I$, while (36), the only other constraint, only puts a lower bound on I.

B.9. Proof of Theorem 2

Step 1. First, we show that $\Gamma(q,\kappa_0,\Delta\kappa)$ is strictly decreasing in q and strictly increasing in $\Delta\kappa$. By definition of $\bar{K}(q)$, we have

$$(r-s)\int_{\bar{d}(\bar{K}(q))}^{q}\left(\bar{F}_1-\bar{F}_0\right)(u)\,du=\Delta\kappa$$

Taking the first derivative with respect to q gives

$$\left(\bar{F}_1-\bar{F}_0\right)(q)-\frac{\partial\bar{d}\left(\bar{K}(q)\right)}{\partial q}\left(\bar{F}_1-\bar{F}_0\right)\left(\bar{d}\left(\bar{K}(q)\right)\right)=0\quad\text{or}\quad\frac{\partial\bar{d}\left(\bar{K}(q)\right)}{\partial q}=\frac{\left(\bar{F}_1-\bar{F}_0\right)(q)}{\left(\bar{F}_1-\bar{F}_0\right)\left(\bar{d}\left(\bar{K}(q)\right)\right)}.$$

Given that $\bar{K}(q)>sq$, $\bar{d}\left(\bar{K}(q)\right)=\left(\bar{K}(q)-sq\right)/(r-s)$ and $\partial\bar{K}/\partial q(q)=(r-s)\,\partial\bar{d}/\partial q\left(\bar{K}(q)\right)+s$. Now since

$$\Gamma(q,\kappa_0,\Delta\kappa)=\left(rq-\bar{K}(q)\right)-(r-s)\int_{\bar{d}(\bar{K}(q))}^{q}F_1(u)\,du-\kappa_1$$

we have

$$\begin{aligned}
\frac{\partial\Gamma(q,\kappa_0,\Delta\kappa)}{\partial q}&=r-\frac{\partial\bar{K}(q)}{\partial q}-(r-s)F_1(q)+(r-s)\frac{\partial\bar{d}\left(\bar{K}(q)\right)}{\partial q}F_1\left(\bar{d}\left(\bar{K}(q)\right)\right)\\
&=r-\left((r-s)\frac{\partial\bar{d}\left(\bar{K}(q)\right)}{\partial q}+s\right)-(r-s)F_1(q)+(r-s)\frac{\partial\bar{d}\left(\bar{K}(q)\right)}{\partial q}F_1\left(\bar{d}\left(\bar{K}(q)\right)\right)\\
&=(r-s)\left(\bar{F}_1(q)-\frac{\partial\bar{d}\left(\bar{K}(q)\right)}{\partial q}\bar{F}_1\left(\bar{d}\left(\bar{K}(q)\right)\right)\right)=(r-s)\left(\bar{F}_1(q)-\frac{\left(\bar{F}_1-\bar{F}_0\right)(q)}{\left(\bar{F}_1-\bar{F}_0\right)\left(\bar{d}\left(\bar{K}(q)\right)\right)}\bar{F}_1\left(\bar{d}\left(\bar{K}(q)\right)\right)\right)\\
&=\frac{(r-s)}{\left(\bar{F}_1-\bar{F}_0\right)\left(\bar{d}\left(\bar{K}(q)\right)\right)}\left(\bar{F}_1(q)\left(\bar{F}_1-\bar{F}_0\right)\left(\bar{d}\left(\bar{K}(q)\right)\right)-\left(\bar{F}_1-\bar{F}_0\right)(q)\bar{F}_1\left(\bar{d}\left(\bar{K}(q)\right)\right)\right)\\
&=\frac{(r-s)}{\left(\bar{F}_1-\bar{F}_0\right)\left(\bar{d}\left(\bar{K}(q)\right)\right)}\left(\bar{F}_0(q)\bar{F}_1\left(\bar{d}\left(\bar{K}(q)\right)\right)-\bar{F}_1(q)\bar{F}_0\left(\bar{d}\left(\bar{K}(q)\right)\right)\right)\\
&=\frac{(r-s)\bar{F}_0(q)\bar{F}_0\left(\bar{d}\left(\bar{K}(q)\right)\right)}{\left(\bar{F}_1-\bar{F}_0\right)\left(\bar{d}\left(\bar{K}(q)\right)\right)}\left(\frac{\bar{F}_1\left(\bar{d}\left(\bar{K}(q)\right)\right)}{\bar{F}_0\left(\bar{d}\left(\bar{K}(q)\right)\right)}-\frac{\bar{F}_1(q)}{\bar{F}_0(q)}\right)
\end{aligned}$$

which is strictly negative given that $\bar{F}_1/\bar{F}_0$ is strictly increasing (from MLRP) and $\bar{d}\left(\bar{K}(q)\right)<q$ since $K<rq$.

Moreover, (B.9) implies that as $\Delta\kappa$ increases, $\bar{d}(K)$ and hence $\bar{K}(q)$ must decrease strictly, and so $\mathbb{E}\left[\bar{K}(q)\wedge P_{0,q}\right]$ must also decrease strictly. Further, by definition of $\Gamma(q,\kappa_0,\Delta\kappa)$ and $\bar{K}(q)$, we have

$$\Gamma(q,\kappa_0,\Delta\kappa)=\mathbb{E}\left[P_{1,q}-\bar{K}(q)\wedge P_{1,q}\right]-\kappa_1=\mathbb{E}\left[P_{0,q}-\bar{K}(q)\wedge P_{0,q}\right]-\kappa_0$$

Hence

$$\frac{\partial\Gamma(q,\kappa_0,\Delta\kappa)}{\partial\Delta\kappa}=-\frac{\partial\mathbb{E}\left[\bar{K}(q)\wedge P_{0,q}\right]}{\partial\Delta\kappa}>0$$

Step 2. Next, assume $\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa^{\max}\right)\le 0$. We have

$$\left(\pi_1\left(q_1^{FB}\right)-\kappa_1\right)-\left(\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa\right)\right)^+=\left(\pi_1\left(q_1^{FB}\right)-\kappa_1\right)>0$$

Hence (48) holds. Moreover, for all $q\in[\hat{q},\bar{q}_1]$ with $q\neq q_1^{FB}$ we have

$$\left(\pi_1(q)-\kappa_1\right)-\left(\Gamma(q,\kappa_0,\Delta\kappa)\right)^+\le\left(\pi_1(q)-\kappa_1\right)<\left(\pi_1\left(q_1^{FB}\right)-\kappa_1\right)$$

Hence by (49), optimality implies $q^*=q_1^{FB}$.

Step 3. Now assume $\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa^{\max}\right)>0$. Because $\lim_{\Delta\kappa\to 0}\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa\right)=-\kappa_0<0$and $\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa\right)$ is strictly increasing in $\Delta\kappa$, we can define a unique threshold $\Delta\underline{\kappa}$ by$\Gamma\left(q_1^{FB},\kappa_0,\Delta\underline{\kappa}\right)=0$. For all $\Delta\kappa\in(0,\Delta\underline{\kappa}]$, $\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa\right)\le 0$ and as per the previous reasoning, the first-best obtains: $q^*=q_1^{FB}$.

Consider now $\kappa_1\in(\Delta\underline{\kappa},\Delta\kappa^{\max})$. Note that $\lim_{\Delta\kappa\to\Delta\kappa^{\max}}\left(\pi_1\left(q\right)-\left(\kappa_0+\Delta\kappa\right)\right)-\left(\Gamma\left(q,\kappa_0,\Delta\kappa\right)\right)^+<0$. Indeed, $\forall q>q_1^{FB}$ the first term is strictly negative for $\Delta\kappa\to\Delta\kappa^{\max}$ by definition of q_1^{FB} and for $q=q_1^{FB}$, the first term equals zero but $\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa\right)>0$ as by definition of $\Delta\underline{\kappa}$. Besides, the first term decreases strictly with $\Delta\kappa$ and $\Gamma\left(q,\kappa_0,\Delta\kappa\right)$ increases strictly with $\Delta\kappa$. Hence we can define a unique $\Delta\bar{\kappa}<\Delta\kappa^{\max}$ as the minimum $\Delta\kappa$ such that $\forall q\ge q_1^{FB}$, $\left(\pi_1\left(q\right)-\left(\kappa_0+\Delta\kappa\right)\right)-\left(\Gamma\left(q,\kappa_0,\Delta\kappa\right)\right)^+\le 0$. For $\Delta\kappa\in(\Delta\bar{\kappa},\Delta\kappa^{\max})$, the problem does not have a solution and the project is abandoned ($q^*=0$).

Note that $\Delta\underline{\kappa}<\Delta\bar{\kappa}$, i.e., $\exists q$ exists such that $\left(\pi_1\left(q\right)-\left(\kappa_0+\Delta\underline{\kappa}\right)\right)-\left(\Gamma\left(q,\kappa_0,\Delta\underline{\kappa}\right)\right)^+>0$. Indeed, the condition holds for $q=q_1^{FB}$: the first term is $\pi_1\left(q_1^{FB}\right)-\kappa_1>0$ and the second term is zero.

Last, we show that for $\Delta\kappa\in(\Delta\underline{\kappa},\Delta\bar{\kappa})$, the optimal capacity exceeds q_1^{FB}. For $\Delta\kappa$ such that

$$\left(\pi_1\left(q_1^{FB}\right)-\left(\kappa_0+\Delta\kappa\right)\right)-\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa\right)<0$$

the optimum cannot equal q_1^{FB} and must therefore exceed q_1^{FB}. Consider now $\Delta\kappa$ such that

$$\left(\pi_1\left(q_1^{FB}\right)-\left(\kappa_0+\Delta\kappa\right)\right)-\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa\right)\ge 0$$

Since $\Gamma\left(q_1^{FB},\kappa_0,\Delta\kappa\right)>0$, it must be that the objective is also $\left(\pi_1\left(q\right)-\left(\kappa_0+\Delta\kappa\right)\right)-\Gamma\left(q,\kappa_0,\Delta\kappa\right)$ and is strictly positive in a right-neighborhood of q_1^{FB}. Hence, in that neighborhood, its first derivative with respect to q is

$$\frac{\partial}{\partial q}\left(\pi_1\left(q\right)-\left(\kappa_0+\Delta\underline{\kappa}\right)\right)-\frac{\partial}{\partial q}\Gamma\left(q,\kappa_0,\Delta\underline{\kappa}\right)$$

The first term is zero at $q=q_1^{FB}$ by definition of q_1^{FB}. The second term is strictly negative. Hence the derivative is strictly positive at $q=q_1^{FB}$. Therefore the objective is strictly positive in a right-neighborhood of q_1^{FB} and strictly increasing at q_1^{FB}. Hence $q^*>q_1^{FB}$.